

# A MULTIDIMENSIONAL ANALOGUE OF THE DARBOUX PROBLEM FOR HYPERBOLIC EQUATIONS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## A MULTIDIMENSIONAL ANALOGUE OF THE DARBOUX PROBLEM FOR HYPERBOLIC EQUATIONS

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Let  $D$  be a simply connected domain of three-dimensional Euclidean space  $E_3$  of points  $(x_1, x_2, t)$ , bounded by two surfaces  $t = |x|$ ,  $t = r - |x|$ , where  $|x|^2 = x_1^2 + x_2^2$ ,  $r = \text{const} > 0$ , situated in the half-space  $x_2 \leq 0$ , and by the plane  $x_2 = 0$ ; let  $S_1$ ,  $S_2$ , and  $S_3$  be the parts of the surfaces  $t = |x|$ ,  $t = r - |x|$ , and  $x_2 = 0$ , respectively, forming the boundary  $\partial D$  of the domain  $D$ .

In the domain  $D$  consider the equation

$$Lu \equiv u_{x_1 x_1} + u_{x_2 x_2} - u_{tt} + a^i u_{x_i} + b u_t + c u = f \quad (1)$$

with coefficients  $a^i = a^i(x, t)$ ,  $i = 1, 2$ ,  $b = b(x, t)$ ,  $c = c(x, t)$  of class  $C^2(\overline{D})$  and right-hand side  $f = f(x, t)$  from the space  $L_2(D)$ . Here and below, repetition of the index  $i$  denotes summation over it from 1 to 2.

Equation (1), by the change of dependent variable according to the formula  $u = u_1 \exp(\mu t)$ , where  $\mu = \text{const} > 0$ , can be transformed into an equation (with respect to  $u_1$ ) with the coefficient at  $u_1$  everywhere in the closure  $\overline{D}$  less than any prescribed negative number. Therefore, without loss of generality, we shall assume that

$$c < -\max_{\overline{D}} |a_{x_i}^i + b_t|. \quad (2)$$

The analogues of the (mixed) Darboux problem <sup>(1)</sup> are the following two problems.

**Problem B.** Find a solution  $u(x, t)$  of equation (1) satisfying the boundary conditions

$$u|_{S_2} = 0, \quad u|_{S_3} = 0. \quad (\text{B})$$

**Problem B<sup>+</sup>.** Find a solution  $v(x, t)$  of the equation

$$L^+x \equiv v_{x_1x_1} + v_{x_2x_2} - v_{tt} - (a^i v)_{x_i} - (bv)_t + cv = f, \quad (1+)$$

satisfying the boundary conditions

$$v|_{S_1} = 0, \quad v|_{S_3} = 0.$$

These problems for the two-dimensional wave equation were first formulated by A. V. Bitsadze <sup>(2)</sup>. They are exceptional cases of a problem studied by S. L. Sobolev <sup>(3)</sup> for the multidimensional wave equation, when the data are prescribed on a time-like oriented conical surface.

Introduce into consideration the Hermitian bilinear form

$$(u, v)_+ = \int_D [\alpha \nabla u \nabla v + 2(b - \beta)u_t v_t + a^i u_{x_i} v_t + a^i v_{x_i} u_t + uv] dx dt, \quad (3)$$

where  $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial t)$ , and  $\alpha$  and  $\beta$  are arbitrary positive numbers, satisfying the inequality

$$\alpha > \max \left[ \max_{\overline{D}} (1 + |b - \beta| + \sqrt{|b - \beta|^2 + |a|^2}), \max_{\overline{D}} \left| \frac{c_t}{c} \right|, \max_{\overline{D}} |a|^2, \max_{\overline{D}} \left| \frac{c_t - a^i x_t^i - b_{tt}}{c - a^i x_i^i - b_t} \right| \right], \quad |a|^2 = (a^1)^2 + (a^2)^2 + (a^3)^2$$

Since all roots (characteristic numbers) of the characteristic equation

$$(\alpha - \lambda) [(\alpha - \lambda)^2 + 2(b - \beta)(\alpha - \lambda) - |a|^2] = 0,$$

corresponding to the quadratic form

$$\alpha |\xi|^2 + 2(b - \beta)\xi_3^2 + 2a^i \xi_i \xi_3, \quad (x, t) \in \overline{D}, \quad \xi \in E_3,$$

are positive by virtue of (14), it follows that

$$(u, u)_+ \geq 0, \quad (u, u)_+ = 0 \iff u = 0. \quad (5)$$

Consequently, for the form (3) all the axioms of an inner product are satisfied.

We shall adopt the following notation:

$w_2^k(D)$  is the positive Sobolev space with norm  $\|\cdot\|_k$  and inner product  $(\cdot, \cdot)_k$ ,  $w_2^0(D) = L_2(D)$ .

$H_+$  is the Hilbert space of functions summable in the domain  $D$ , having generalized derivatives up to first order with finite norm defined by the inner product (3):  $\|u\|_+^2 = (u, u)_+$ ; the completeness of the space  $H_+$  follows from the estimate  $\|u\|_1 \leq \text{const}\|u\|_+$ ; evidently,  $\|u\|_0 \leq \|u\|_+$ .

$H_-$  is the negative (Hilbert) space constructed from the zero space  $H_0 = L_2(D)$  and the positive space  $H_+$ , i.e. the completion of the space  $H_0$  in the norm

$$\|\varphi\|_- = \sup_{u \in H_+} \frac{|(\varphi, u)_0|}{\|u\|_+} \leq \|\varphi\|_0.$$

$w_2^{-1}(D)$  is the negative Sobolev space constructed from  $H_0$  and  $w_2^1(D)$ .

$\langle \varphi, u \rangle_0$ , where  $u \in H_+$ , is a bilinear form on the space  $H_-$ , which for  $\varphi \in H_0$  coincides with the inner product  $(\varphi, u)_0$  in  $H_0$ . It is known (see, for example, (4)) that for this form the generalized Schwarz inequality holds:

$$|\langle \varphi, u \rangle_0| \leq \|\varphi\|_- \|u\|_+. \quad (6)$$

$w$  and  $w^+$  are the sets of all functions of the class  $C(\overline{D}) \cap C^2(D) \cap w_2^1(D) \cap w_2^1(\partial D)$  for which  $Lu \in H_0$  and the conditions B and  $B^+$ , respectively, are satisfied.

A weak solution of problem B will be called any function  $u \in L_2(D)$  satisfying the equality

$$(u, L^+v)_0 = (f, v)_0, \quad \forall v \in w^+. \quad (7)$$

Below we shall assume that the right-hand side  $f$  belongs to the space  $H_- \supseteq L_2(D)$ .

**A priori estimate.** For any functions  $u \in w$  and  $v \in w^+$ , the inequalities

$$\|u\|_+ \leq C\|Lu\|_0, \quad \|v\|_+ \leq C\|L^+v\|_0, \quad (8)$$

hold, where  $C$  is a positive constant independent of  $u$  and  $v$ .

Indeed, for any function  $u \in w$  and  $\bar{d} = d(t)$  of class  $C^1(\overline{D})$  the identity

$$\begin{aligned} 2(du_t, L_1u)_0 &= \int_D [d_t|\nabla u|^2 + 2dbu_t^2 + 2da^i u_{x_i} u_t - (dc)_t u^2] dx dt \\ &+ \int_{\partial D} dc n^3 u^2 dS + \int_{\partial D} d(-|\nabla u|^2 n^3 + 2n^i u_{x_i} u_t) dS = I_1 + I_2 + I_3, \end{aligned} \quad (9)$$

holds, where  $n^i = \cos(\widehat{n, x_i})$ ,  $i = 1, 2$ ,  $n^3 = \cos(\widehat{n, t})$  are the direction cosines of the exterior normal  $n = (n^1, n^2, n^3)$  to  $\partial D$ ;  $dS$  is the surface element of  $\partial D$ .

Let  $d = \exp(\alpha t)$ . Then, by virtue of (2) and the inequality  $n^3 \leq 0$  on  $S_1$ , it is clear that  $I_2 \geq 0$ . Since on the surface  $S_3$ ,  $n = (0, 1, 0)$ , on the basis of condition (B) one may write

$$I_3 = \int_{S_1 \cup S_2} d \left( -|\nabla u|^2 n^3 + 2n^i u_{x_i} u_t \right) dS = I_3(S_1) + I_3(S_2).$$

On the characteristic  $S_2$  of equation (1) we have

$$(n^1)^2 + (n^2)^2 = (n^3)^2, \quad u = 0 \Rightarrow u_{x_i} = u_n n^i, \quad u_t = u_n n^3.$$

Consequently, on  $S_2$

$$-|\nabla u|^2 n^3 + 2n^i u_{x_i} u_t = n^3 u_n^2 [(n^1)^2 + (n^2)^2 - (n^3)^2] = 0,$$

therefore  $I_3(S_2) = 0$  for any function  $\bar{d}$ .

Further, taking into account that on the characteristic  $S_1$ :  $\sqrt{(n^1)^2 + (n^2)^2} = -n^3$ , all roots of the characteristic equation

$$\lambda(\lambda + n^3)(\lambda + 2n^3) = 0,$$

corresponding to the quadratic form

$$-n^3 |\xi|^2 + 2n^i \xi_i \xi_3, \quad (x, t) \in S_1, \quad \xi \in E_3,$$

are nonnegative, we obtain  $I_3(S_1) \geq 0$ ,  $\forall d \geq 0$ .

From (9), taking into account that  $d_t = \alpha d$  and

$$2(du_t, Lu)_0 \leq 2\beta \|\sqrt{d} u_t\|_0^2 + C_1 \|Lu\|_0^2,$$

where  $C_1$  is a positive constant independent of  $u$ , we have

$$\int_D \bar{d} [\alpha |\nabla u|^2 + 2(b - \beta) u_t^2 + 2a^i u_{x_i} u_t] dx dt - \int_D (dc)_t u^2 dx dt \leq C_1 \|Lu\|_0^2.$$

Hence, from inequalities (2) and (4), we obtain the first a priori estimate in (8).

The second a priori estimate in (8) is proved in a completely analogous way, if, as the auxiliary function  $d$ , one takes the function  $d = -\exp(-\alpha t)$ .

It is easy to see that, for the above-indicated values of  $\alpha$ , the norms  $\|\cdot\|_+$  and  $\|\cdot\|_1$  are equivalent; therefore, from inequality (9) follow the a priori estimates

$$\|u\|_1 \leq C\|Lu\|_0, \quad \|v\|_1 \leq C\|L^+v\|_0, \quad \forall u \in w, v \in w^+. \quad (10)$$

Inequalities (10) generalize the a priori estimate

$$\|u_{x_2}\|_0 \leq C\|\square u\|_0, \quad \forall u \in w,$$

where  $\square = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 - \partial^2/\partial t^2$  is the Lorentz operator, obtained by A. V. Bitsadze <sup>(2)</sup>.

By integration by parts one can show that

$$(u, L^+v)_0 = (Lu, v)_0, \quad \forall u \in w, v \in w^+,$$

therefore the problems B and B<sup>+</sup> are (formally) adjoint.

From the a priori estimate (10) follows the uniqueness of a regular (classical) solution  $u \in w$  or  $v \in w^+$  of the problems B and B<sup>+</sup>. For  $L = \square$  this fact was first established in <sup>(2)</sup>.

The proof of existence of a weak solution of problem B is carried out according to the standard scheme, which we reproduce for ease of reading. According to the a priori estimates (8), the expression  $\langle f, v \rangle_0$  from (7) depends not on  $v$ , but on  $L^+v$ , and therefore one may put  $\langle f, v \rangle_0 = F(L^+v)$ , where  $F$  is a single-valued and additive functional on the linear set  $L^+(w^+)$ , where  $L^+(w^+)$  is the image of  $w^+$  under the mapping  $v \rightarrow L^+v$ . Further, on the basis of (6) and (8), we have

$$|F(L^+v)| = |\langle f, v \rangle_0| \leq \|f\|_- \|v\|_+ \leq C\|f\|_- \|L^+v\|_0,$$

i.e., for fixed  $f$  the functional  $F(\psi)$ ,  $\psi = L^+v$ , on  $L^+(w^+)$  is continuous. Extending  $F(\psi)$ , by the well-known Hahn–Banach theorem, to the whole space  $L_2(D)$  and using the Riesz theorem, we find the desired function  $u$ :  $F(\psi) = (u, \psi)_0$ ,  $\forall \psi \in L_2(D)$ , and, in particular, for  $\psi = L^+v$ ,

$$(u, L^+v)_0 = F(L^+v) = \langle f, v \rangle_0.$$

The energy inequalities (10) ensure the existence of a weak solution for any right-hand side  $f$  from  $w_2^{-1}(D)$ . If, however,  $f \in L_2(D)$ , then, as is known (see, for example, <sup>(4)</sup>), they also guarantee the existence of a semistrong solution. We also note that a priori estimates of the form (10) hold if the condition  $u|_{s_3} = 0$  is replaced by the condition  $u_{x_2}|_{s_3} = 0$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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