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Abstract

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MATHEMATICS

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ALMOST-PERIODIC SOLUTIONS OF LINEAR AND NONLINEAR EQUATIONS IN A BANACH SPACE

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1. Introduction.

In a separable Banach space B we consider the equation

$$u' = Q(t)u, \quad (1)$$

where $Q(t)$ is, in general, an unbounded nonlinear operator depending on t almost periodically. Beginning with Favard's work ⁽¹⁾, the conditions for the existence of almost-periodic (a.p.) solutions are formulated in terms of the so-called limiting equations.

Suppose that the domain of definition $D = D(Q(t))$ does not depend on t and is a separable Banach space dense in B . Let $Q(t)$ be a continuous transformation $B \rightarrow B$, depending on t almost periodically uniformly on each bounded subset D . Then from every sequence $\{t_m\}$ one can extract a subsequence $\{t'_m\}$ such that $Q(t + t'_m)u \rightarrow Q(t)u$. The set of elements h of the form $h = Q(s)$ forms a compact metric group G with a system of shifts L_t and an invariant measure λ . The limiting equations will be equations of the form

$$u' = \hat{Q}(t)u. \quad (1')$$

Below we shall use (in general, separately) the following assumptions concerning equations (1'):

- a) the Cauchy problem is solvable for every initial value $u \in B$, and the resolving operator $T_h(t)$ is strongly continuous for $t \geq 0$;
- b) the resolving operator $T_h(t)$ is weakly continuous for $t \geq 0$;

- c) a condition of continuous dependence on the “right-hand sides” is satisfied, namely: if $\|u\| \leq c < \infty$, $h_m \rightarrow h$, then $\|T_{h_m}(t)u - T_h(t)u\| \rightarrow 0$ for $t > 0$ uniformly in u .

Further, if (1) is a linear equation of the form $u' = A(t)u + f(t)$, then it is more convenient to use the resolving operator $V_h(t)$ of the corresponding homogeneous problems. Here condition b) follows from a).

Put $X = B \times G$ and $S_t x = S_t\{u, h\} = \{T_h(t)u, L_t h\}$. It is easy to verify (see (3–5)) that the transformations S_t form a dynamical system if B is endowed with the corresponding (strong or weak) topology.

2. A.p. in the sense of Besicovitch solutions of linear equations.

In the works of S. L. Sobolev ⁽⁶⁾, L. Amerio ⁽¹⁰⁾, and others (see ⁽⁹⁾), the question of a.p. solutions of hyperbolic equations was studied; the methods of investigation rely on the existence (for the homogeneous equation) of the energy integral. The results of Favard’s abstract theory ^(7,8) make it possible to study the case of energy dissipation, but no more than that. In the general case of variable energy, the problems of almost-periodicity can be solved only in the Besicovitch class. The corresponding theorem is formulated for a first-order equation in a Hilbert space.

Theorem 1. Let A be the generator of a strongly continuous semigroup, $B(t)$ an a.p. in the sense of the operator norm function, and $f(t)$ an a.p. function. Then, if the equation $u' = Au + B(t)u + f(t)$ has a bounded solution, there exists a set $\Lambda \subset G$ of full λ -measure such that, for $h \in \Lambda$, the corresponding equation $u' = Au + \hat{B}(t)u + \hat{f}(t)$ has

i.e., in the sense of Bohr, the solution has the same module of Fourier exponents as $\{B(t), f(t)\}$.

Remark 1. Let $u(t)$ be a bounded solution and let $X_0 \subset X$ be the closure of the corresponding trajectory. Denote by X_1 the closed invariant subset such that $X_0 \subset X_1$ and the layer over some element $h \in G$ is a convex set in H . Then the solutions furnished by Theorem 1 may be chosen from X_1 .

Remark 2. For the hyperbolic equation

$$u'' + A^2 u + B(t)u = f(t)$$

the conditions of Theorem 1 are fulfilled if A is a positive definite operator and $A^{-1}B(t)$ is a.p. in the sense of the operator norm of functions. Here boundedness of the solution means boundedness of the “energy” $\|Au\|^2 + \|u'\|^2$.

Remark 3. Theorem 1 generalizes to the case when A depends periodically on t and the usual solvability conditions for the Cauchy problem are satisfied. The general case, when the principal part A depends on t almost periodically, presents difficulties.

3. General theorems on a.p. solutions of nonlinear equations. Suppose that equations (2) possess the property of uniform positive stability, i.e., the family of transformations $T_h(t)$ ($t \geq 0, h \in G$) is strongly uniformly continuous. This condition alone is apparently quite insufficient for selecting a.p. solutions. Below two special forms of stability are considered; with one of them the method of contracting shift operators is connected, with the other—Favard-type theorems of the first kind.

Theorem 2. Let conditions a, c be satisfied and

$$\varphi(T_h(t)u_1, T_h(t)u_2) \leq \varphi(u_1, u_2) \quad (t \geq 0, u_1, u_2 \in B), \quad (2)$$

where $\varphi(u_1, u_2)$ is strongly continuous on $B \times B$ and satisfies the conditions:

- 1) $\varphi(u_1, u_2) = 0$, if and only if $u_1 = u_2$,
- 2) $\varphi(u_1, \alpha_1 u_1 + \alpha_2 u_2) + \varphi(\alpha_1 u_1 + \alpha_2 u_2, u_2) = \varphi(u_1, u_2)$, where $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$,
- 3) $\varphi(u_1, p) + \varphi(p, u_2) > \varphi(u_1, u_2)$, if $p \notin [u_1, u_2]$.

Then, if equation (1) has a solution compact for $t \geq 0$, it has at least one a.p. solution.

Let us note that, in applications, as the function φ one usually takes $\|u_1 - u_2\|$, provided only that the norm is strictly convex.

Theorem 3. Let inequality (2) be satisfied for a function $\varphi(u_1, u_2)$ with properties 1), 2), 3), and let equation (1) have a weakly compact solution for $t \geq 0$. Then, if the operators $T_h(t)$ are weakly continuous for every $t \geq 0$, equation (1) has at least one a.p. solution.

Theorem 3 reduces to Theorem 2 by proving the existence of a compact solution. This is achieved with the aid of the following lemma (see (8)).

Lemma. If the operators $T_h(t)$ ($t \geq 0, h \in G$) are weakly continuous, condition c_1 is satisfied, and

$$\sup_{t \geq 0} \|T_h(t)u_1 - T_h(t)u_2\| \leq l \|u_1 - u_2\| \quad (l < \infty, h \in G),$$

then every weakly recurrent solution is compact.

Considering, instead of equation (1), powers of a single transformation T , we obtain new criteria for the existence of a fixed point of a contraction of a Banach space with strictly convex norm. It is assumed that T is defined on all of B and that the uniqueness condition is satisfied, i.e., $Tu_1 \neq Tu_2$ if $u_1 \neq u_2$.

Theorem 4. A contraction T has at least one fixed point if there exists an element $u_0 \in B$ for which the set $\{T^m u_0\}$, where $m \geq 0$, is compact. If, however, T is weakly continuous, then weak compactness of the same set is sufficient.

We now formulate the “strong” version of Favard’s first theorem.

Theorem 5. Suppose that conditions a, c, and d are satisfied for every $M > 0$:

$$\lim_{t \rightarrow +\infty} \sup_{h \in G, \|u_1\|, \|u_2\| \leq M} \|T_h(t)u_1 - T_h(t)u_2\| = 0. \quad (3)$$

Then, if equation (1) has a solution bounded for $t \geq 0$, it has a unique a.p. solution.

Let us note that if equation (1) is linear, then the condition of the existence of a bounded solution is superfluous—it follows from (3). There is also a more general assertion for linear equations.

Theorem 6. If B is reflexive, the transformations $V_h(t)$ ($t \geq 0$, $h \in G$) are weakly uniformly continuous on the unit ball, and the homogeneous equation (1) has no bounded solutions, then equation (1) has a unique weak a.p. solution.

4. Applications. In a Banach space B with strictly convex dual B^* , consider the equation

$$u' + Au = f(t), \quad (4)$$

where A is monotone in the sense of Kato ⁽¹²⁾, $\sup_{t \in I} \|f(t)\| < \infty$, and $f(t)$ is an a.p. function in $L_2(0, 1, B)$. Put $\Phi_1(u) = \operatorname{Re}\langle Au, F(u) \rangle$, $\Phi_2(u_1, u_2) = \operatorname{Re}\langle Au_1 - Au_2, F(u_1 - u_2) \rangle$, where $F : B \rightarrow B^*$, $\|F(u)\| = \|u\|$, $\langle u, F(u) \rangle = \|u\|^2$.

Assume that the condition

$$\lim_{s \rightarrow \infty} 1/s \inf_{\|u\| \geq s} \Phi_1(u) = +\infty$$

is satisfied.

Theorem 7. Suppose there is a Banach space $B_1 \subset B$ with completely continuous embedding and the estimate

$$\Phi_1(u) \geq C_1 \|u\|_1^2 - C_2 \|u\|^2 \quad (C_1, C_2 > 0).$$

Then equation (4) has at least one a.p. solution.

Theorem 8. If the inequality

$$\Phi_2(u_1, u_2) \geq C(R) \|u_1 - u_2\|^2 \quad (\|u_1 - u_2\| \leq R, R, C(R) > 0),$$

holds, then equation (4) has a unique a.p. solution. This solution satisfies equation (4) in the ordinary sense if $f(t)$ satisfies the Lipschitz condition $\|f(t_1) - f(t_2)\| \leq L|t_1 - t_2|$.

Theorems 7 and 8 are also valid in the case where A depends on t , but the formulations are too cumbersome.

Let us now consider, in a Hilbert space, the equation

$$u'' + B(u') + A^2 u = f(t), \quad (5)$$

where A is a positive definite operator, $\sup_{t \in I} \|f(t)\| < \infty$, $f(t)$ is an a.p. function in $L_2(0, 1, H)$, and the nonlinear operator B satisfies the following conditions:

1) $D(B) \supseteq D(A^m)$ for some $m > 0$, $(Bu_1 - Bu_2, u_1 - u_2) \geq 0$ for $u_1, u_2 \in D(B)$;

2) the operator $A^{-m}BA^{-1}$ extends to a continuous transformation $H \rightarrow H$.

Denote by H_A the Hilbert space of elements $u \in D(A)$ with norm $\|u\|_A = \|Au\|$. In the energy space $E = H \times H_A$, equation (5) reduces to the equation $z' + \tilde{A}z = \tilde{f}(t)$, where the operator $\tilde{A}$, after a certain closure (or extension) procedure, is monotone in the sense of Kato. Starting from Kato's solutions, generalized solutions are constructed for every $z(0) \in E$.

Assume further that there exists a reflexive Banach space $D(A^n) \subseteq B_1 \subseteq H$ with norm $\|u\|_1$, such that the operator $A^{-1}B$ extends to ...

of a bounded and continuous transformation $K : B_1 \rightarrow H$, and on bounded subsets of B_1 the operator K acts H_A -continuously into H .

Let

$$a_1(s) = \inf_{\|u\|_1 \geq s} (Bu, u), \quad a_2(s) = \inf_{\|u_1 - u_2\|_1 \geq s} (Bu_1 - Bu_2, u_1 - u_2).$$

$$\|Ku\| \leq \beta_1(\|u\|_1)\beta_2(\|u\|), \quad \|Ku_1 - Ku_2\| \leq \gamma_1(\|u_1 - u_2\|_1)\gamma_2(\|u_1 - u_2\|),$$

where the functions β_i, γ_i ($i = 1, 2$) are continuous on $[0, \infty)$.

Theorem 9. Let

$$\lim_{s \rightarrow \infty} a_1/s = +\infty, \\ \lim_{s \rightarrow \infty} \beta_1/a_1 < \infty, \quad \lim_{s \rightarrow \infty} \beta_2/s = 0.$$

Then for every solution $u(t)$ the inequality

$$\sup_{t \in I} \|u(t)\|_E^2 \leq \|u(0)\|_E^2 + C$$

holds, where C does not depend on the solutions. If, moreover, the embedding $B_1 \rightarrow H$ is completely continuous, then equation (5) has at least one E -a.p. solution.

Theorem 10. Let

$$\lim_{s \rightarrow \infty} a_1/s = +\infty,$$

$$\lim_{s \rightarrow \infty} \beta_1/a_1 < \infty, \quad \lim_{s \rightarrow \infty} \beta_2/s = 0,$$

$$\lim_{s \rightarrow \infty} \gamma_1/a_2 < \infty, \quad \lim_{s \rightarrow \infty} \gamma_2/s = 0.$$

Then, if the operator A^{-1} is completely continuous, $a_2(s) \neq 0$ for $s \neq 0$, and the function $f(t)$ is uniformly continuous on the entire axis, equation (5) has a unique E -a.p. solution. This solution satisfies equation (5) in the ordinary sense if $B_1 \subset H_A$, $a_2(s) \geq Cs^2$ ($C > 0$, $0 \leq s \leq 1$), and $f(t)$ satisfies the Lipschitz condition.

Let us note that the delicate results of Prouse follow easily from Theorem 10⁽¹¹⁾. We also note that if $f(t)$ is a periodic function, then the solutions given by Theorems 7-10 have the same period. There is no doubt that these results on periodic solutions (given, for example, by Theorem 9 for a Euclidean space) are difficult to obtain on the basis of the known fixed-point principles.

Note added in proof. We make two remarks concerning Favard's theory.

1. Let R be a complete metric space with metric ρ , and let S_t be a system on $R \times H$ in the sense of item 1. The following "contraction mapping principle" holds (for comparison see Theorems 5 and 8): if for some $t_0 > 0$ the inequality

$$\rho(S_h(t_0)u_1, S_h(t_0)u_2) \leq q\rho(u_1, u_2) \quad (q < 1, h \in G)$$

is satisfied, then equation (1) has a unique a.p. solution.

2. Let B be an arbitrary Banach space, and let equation (1) be linear and possess the property of uniform positive stability, i.e.

$$\sup_{t \geq 0, h \in G} \|V_h(t)\| \leq l < \infty.$$

Then equation (1) has an a.p. solution if and only if it has a weakly compact solution (proof in⁽⁸⁾). This assertion is more general than that which follows from Theorem 3.

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Note: Figure translations are in progress. See original paper for figures.

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