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Abstract

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MATHEMATICS

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ON THE DECOMPOSITION OF A FIELD OF SETS OBTAINED BY AN R -OPERATION OVER RECURSIVE SETS

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According to the well-known analogy (Kuznetsov—Addison) that exists between the hierarchies of descriptive set theory and the hierarchies of the theory of recursive functions, Suslin's theorem on the decomposition of the maximal field of sets* obtained by the A -operation over open sets into the sum of the classes of the Borel hierarchy⁽¹³⁾ corresponds to Kleene's theorem on the decomposition of the maximal field of sets obtained by the A -operation over general recursive sets by means of the hyperarithmetic hierarchy^(4,5). Generalizing the concept of the A -operation, A. N. Kolmogorov introduced the class of R -operations, and A. A. Lyapunov, considering the question of the corresponding generalization of Suslin's theorem, established⁽⁷⁻⁹⁾ that for sufficiently powerful R -operations the maximal field T of sets obtained by an R -operation over open sets is strictly broader than the sum of the classes of the transfinite hierarchy beginning with the same open sets and obtained by a finite or countable repetition of operations weaker than the given R -operation. Thus, the problem of decomposing the field T cannot be solved with the aid of hierarchies of length Ω (Ω is the least uncountable ordinal) and still remains unsolved.

The analogue of this problem in the theory of recursive functions is the problem of decomposing the maximal field T_{rec} of sets obtained by a certain R -operation over general recursive sets; this note is devoted to its solution. For each sufficiently powerful R -operation and the corresponding field T_{rec} , we define a hierarchy whose sum of classes is equal to T_{rec} . The length of this hierarchy depends on T_{rec} and may considerably exceed ω_1 (the least nonconstructive ordinal). The hierarchy obtained in the case when the given R -operation is an A -operation is closely connected with Kleene's hyperarithmetic hierarchy, and the above-mentioned Kleene theorem is obtained in this special case from our decomposition theorem.**

1. Indices. We use the terms and notation of⁽¹⁾, § 1. E is an arbitrary nonempty set, called a **space**; $N = \{0, 1, 2, \dots\}$, PN is the set of all subsets of N ; Φ_M is a δ -operation with base M ($\subseteq PN$). We say that a δ s-operation Φ_{M_1}

is (recursively) more powerful than a δs -operation Φ_{M_2} ($M_1, M_2 \subseteq PN$) if there exists such a (general recursive) function $f(i)$ that

$$\Phi_{M_2}\{E_i\}_i = \Phi_{M_1}\{E_{f(i)}\}_i$$

for any sequence $\{E_i\}_{i \in N}$ of subsets of E . $\Phi_{M'}$ is a δs -operation supplementary to Φ_M ; $\Phi_{R(M)}$ is an R -operation over Φ_M ⁽¹⁾. The set

$$X = \Phi_{R(M)}\{E_i\}_i,$$

where $E_i \subseteq E$, can also be obtained as the result of the following—

* Following Hausdorff ⁽¹³⁾, we call a class of sets a **field of sets** if the sum, difference, and intersection of any two sets of this class belong to the same class.

** The question of generalizing Kleene's theorem by means of R -operations was recently studied also by R. G. Hinman ⁽²⁾; however, the hierarchies constructed by him cover only a proper part of the corresponding fields T_{rec} .

of the transfinite process (9). Suppose that for any $\xi \subseteq N$

$$F(\xi) = \{i : i \in \xi \& \theta_i^{-1}(\xi) \in \hat{M}\}$$

($\theta_i(m)$ and $\hat{M}$, see (1)), and that for all $\alpha < \Omega$ the sets $\xi^{(\alpha)} (\subseteq N)$ are inductively defined:

$$\xi^{(0)} = \xi, \quad \xi^{(\alpha)} = F\left(\bigcap_{\beta < \alpha} \xi^{(\beta)}\right) \quad \text{for } \alpha > 0. \quad (1)$$

Obviously, $\xi^{(\alpha+1)} \subseteq \xi^{(\alpha)}$; the least α such that $\xi^{(\alpha+1)} = \xi^{(\alpha)}$ is called the **stabilization index** of ξ and is denoted $\text{Ind } st_M \xi$. If $\alpha = \text{Ind } st_M \xi$, then $\xi^{(\alpha)}$ is denoted $\text{St}_M \xi$. Suppose that for all $x \in E$, $\xi_x = \{i : x \in E_i\}$. Then

$$X = \Phi_{R(M)}\{E_i\} = \{x : 0 \in \text{St}_M \xi_x\}.$$

If $x \in E \setminus X$, then the least α for which $0 \notin \xi_x^{(\alpha)}$ is denoted $\text{Ind}_M(x/\{E_i\})$; if $x \in X$, then put $\text{Ind}_M(x/\{E_i\}) = \Omega$. $\text{Ind}_M(x/\{E_i\})$ is called the **external index** of the sequence $\{E_i\}$ at the point x .

2. The classes $\mathfrak{R}_M$. Further, the space $E = N^{k+1}$ for various $k \in N$. If for all $i \in N$, $E_i \subseteq N^{k+1}$, and

$$\{\langle m_0, \dots, m_k, i \rangle : \langle m_0, \dots, m_k \rangle \in E_i\}$$

is a general (or primitive-) recursive set, then the sequence $\{E_i\}$ is called general (or primitive-) recursive. The class of all sets $X (\subseteq N^{k+1}, k = 0, 1, 2, \dots)$ such that

$$X = \Phi_{R(M)}\{E_i\},$$

where $\{E_i\}$ is some general-recursive sequence ($E_i \subseteq N^{k+1}$ for some $k \in N$), will be called the class $\mathfrak{R}_M$. The class of complements (with respect to the corresponding N^{k+1}) of all sets of the class $\mathfrak{R}_M$ will be denoted $C\mathfrak{R}_M$; $\mathfrak{R}_M \cap C\mathfrak{R}_M$ will be denoted $B\mathfrak{R}_M$. (The classes $\mathfrak{R}_{n+1}$ from (1) coincide with $\mathfrak{R}_M \cap PN$ for $M = R'_n$.) In $\mathfrak{R}_M$ there is a field of sets containing every field composed of $\mathfrak{R}_M$ -sets.

Let us note that:

A1. The class of all sets $X = \Phi_{R(M)}\{E_i\}$, where $\{E_i\}$ is a primitive-recursive sequence, coincides with $\mathfrak{R}_M$.

A2. There exists an $\mathfrak{R}_M$ -set $U \subseteq N^{k+2}$, universal for all $\mathfrak{R}_M$ -subsets of N^{k+1} . Consequently, $\mathfrak{R}_M \neq C\mathfrak{R}_M$, and the classes $\mathfrak{R}_M$ and $C\mathfrak{R}_M$ are strictly wider than $B\mathfrak{R}_M$.

Put

$$\omega_M = \sup\{\text{Ind } st_M \xi : \xi \text{ is a general-recursive subset of } N\}.$$

For any $M \subseteq PN$: a) the class of stabilization indices of all general-recursive $\xi \subseteq N$ coincides with $\{\alpha : \alpha \leq \omega_M\}$; b) the class of all $\neq \Omega$ external indices of all possible general-recursive sequences coincides with $\{\alpha : \alpha < \omega_M\}$. The proof of the index comparison principle for R_α -sets in (8) (simplified with the aid of (9)) also goes through for $\mathfrak{R}_M$ -sets, as a result of which one obtains

Lemma (on comparison of indices). Let $\{E_i^0\}$ and $\{E_i^1\}$ be two general-recursive sequences of subsets of N^{k+1} , and for $x \in N^{k+1}$ and $r = 0, 1$

$$\alpha_r(x) = \text{Ind}_M(x/\{E_i^r\}).$$

Then, if $\Phi_{R(M)}$ is recursively stronger than $\Phi_{M'}$, and than the operation of summing two sets, then

$$\{x : \alpha_0(x) \leq \alpha_1(x)\} \in \mathfrak{R}_M.$$

Corollary. Under the notation and conditions of the lemma, and for any $\beta < \omega_M$, the sets

$$\{x : \alpha_0(x) < \beta\}, \quad \{x : \alpha_0(x) \leq \beta\}, \quad \{x : \alpha_0(x) = \beta\}$$

belong to $B\mathfrak{R}_M$.

3. Boundedness of indices. Let $\mathfrak{A}$ be the class of all such $M \subseteq PN$ that:

B1. Φ_M is recursively stronger than the operation $\bigcup$ (i.e. $\bigcup_i E_i$).

B2. $\Phi_{R(M)}$ is recursively stronger than $\Phi_{M'}$.

We agree that everywhere below A denotes a variable over $\mathfrak{A}$, and, when composite notation containing A is used (for example, $\mathfrak{R}_A$, $\Phi_{R(A)}$, ω_A), it is always understood that $A \in \mathfrak{A}$.

Theorem 1 (on boundedness of indices). Let $X, Y \subseteq N^{k+1}$, $X \cap Y = \emptyset$, and $X, Y \in \mathfrak{R}_A$. Let $\{E_i\}$ be a general recursive sequence of subsets of N^{k+1} such that

$$\Phi_{R(A)}\{E_i\} = Y.$$

Then there exists an ordinal $\alpha < \omega_A$ such that for all $x \in X$

$$\text{Ind}_A(x/\{E_i\}) < \alpha.$$

The proof is carried out by a method close to that by which in ⁽⁶⁾ Luzin's theorem on boundedness of the indices of a sieve on analytic sets is proved; A2, B1, and the lemma on comparison of indices are used.

One says that a (general recursive) sequence $\{E_i\}$ is a sequence with **bounded external indices** if there exists an $\alpha < \Omega$ ($\alpha < \omega_M$, respectively) such that

$$\text{Ind}_M(x/\{E_i\}) < \alpha$$

for all $x \in \Phi_{R(M)}\{E_i\}$.

Theorem 2. The class of all sets

$$X = \Phi_{R(A)}\{E_i\},$$

where $\{E_i\}$ is any general recursive sequence with bounded external indices, coincides with $\mathfrak{B}\mathfrak{R}_A$.

The proof follows from Theorem 1 and the corollary to the lemma on comparison of indices.

Remark. For the classical R_α -sets with $\alpha > 1$, assertions analogous to Theorems 1 and 2 are false ⁽⁸⁾.

If $X \subseteq N^{k+1}$, $Y \subseteq N$, and $X = f^{-1}(Y)$, where f is a general recursive function all of whose values are distinct, then one writes

$$X \leq_1 Y.$$

Theorem 3. For every $X \in \mathfrak{R}_A$ there exists a general recursive $\xi \subseteq N$ such that

$$X \leq_1 \text{St}_A \xi.$$

For every general recursive $\xi \subseteq N$,

$$\text{St}_A \xi \in \mathfrak{R}_A$$

and

$$(\text{Ind st}_A \xi < \omega_A) \iff (\text{St}_A \xi \in \mathfrak{BR}_A).$$

Proof with the aid of Theorem 2.

We shall now obtain a new characterization of ω_A , not connected with indices. A function $\nu(n)$ mapping some $X \subseteq N$ onto the set of all ordinals $\beta < \alpha$ ($\alpha < \Omega$) is called a **numbering** of the segment $\{\beta : \beta < \alpha\}$. If there exists such a numbering $\nu(n)$ of the segment $\{\beta : \beta < \alpha\}$ for which

$$\{\langle m, n \rangle : \nu(m) < \nu(n)\} \in \mathfrak{BR}_M,$$

then α will be called a $\mathfrak{BR}_M$ -ordinal. If α is a $\mathfrak{BR}_M$ -ordinal, then, evidently, every $\beta < \alpha$ is also a $\mathfrak{BR}_M$ -ordinal.

Theorem 4. ω_A is the least ordinal that is not a $\mathfrak{BR}_A$ -ordinal.

Corollary. If $A_1, A_2 \in \mathfrak{A}$, then $\mathfrak{R}_{A_1} = \mathfrak{R}_{A_2}$ implies

$$\omega_{A_1} = \omega_{A_2}.$$

In particular,

$$\omega_{R(A)} = \omega_A.$$

4. Hierarchies. Let $\varphi(m, n)$ be a general recursive function universal for all one-place primitive recursive functions ⁽¹⁰⁾, and let

$$\pi = \{[m_0, \dots, m_k] : \varphi(m_0, [m_1, \dots, m_k]) = 0\}$$

($[m_0, \dots, m_k]$ is the number of the tuple $\langle m_0, \dots, m_k \rangle$) ⁽¹⁾. From A1 and Theorem 3 it follows that

$$\text{Ind st}_A \pi = \omega_A.$$

Define a numbering $|m|_A$ of the segment $\{\alpha : \alpha < \omega_A\}$ as follows:

$$\{m : |m|_A = \alpha\} = \left(\bigcap_{\beta < \alpha} \pi^{(\beta)} \right) \setminus \pi^{(\alpha)},$$

where $\pi^{(\alpha)}$ are defined by ⁽¹⁾ for $\xi = \pi$ and $M = A$. Let, for any $\xi \subseteq N$,

$$j_A(\xi) = \Phi_A\{E_i^\xi\},$$

where

$$E_i^\xi = \{m : (\exists k) T_1^\xi(m, i, k)\}$$

and T_1^ξ is the known predicate from (3). Now, by induction on $\alpha < \omega_A$, define the sets $H_A(\alpha)$ ($\subseteq N$):

$$H_A(0) = j_A(\emptyset),$$

$$H_A(\alpha) = \{[m, n] : |m|_A < \alpha \text{ and } n \in H_A(|m|_A)\} \quad \text{for } \alpha > 0.$$

The class of all sets X ($\subseteq N^{k+1}$, $k \geq 0$) general recursive relative to $H_A(\alpha)$ will be denoted by $K_A(\alpha)$. Below, $\xi <_1 \eta$ means that $\xi \leq_1 \eta$, but $\eta \not\leq_1 \xi$.

Theorem 5. If $\alpha < \beta < \omega_A$, then

$$H_A(\alpha) <_1 H_A(\beta),$$

and the class $K_A(\beta)$ is strictly wider than $K_A(\alpha)$.

Theorem 6 (decomposition theorem).

$$\mathfrak{B}\mathfrak{R}_A = \bigcup_{\alpha < \omega_A} K_A(\alpha).$$

The inclusion of the right-hand side in the left is proved by effective transfinite induction ⁽¹¹⁾, and the reverse inclusion by means of Theorem 2.

5. Connection with the hyperarithmetical hierarchy

If Φ_A is the operation $\bigcup$ (i.e., $\hat{A} = PN \setminus \{\emptyset\}$), then for every $X \subseteq N^{k+1}$, $X \in \mathfrak{R}_A \iff X \in \Sigma_1^1$, and $\omega_A = \omega_1$ (12).

Let O be the class of all Kleene ordinal notations (5), let $|a|$ be the ordinal denoted by the notation $a \in O$, and let $a^* = 2^a$. Let $H_a = \{x : H_a(x)\}$, where $a \in O$ and $H_a(x)$ are the predicates defined in (4).

Theorem 7. If Φ_A is the operation $\bigcup$, then for every $a \in O$,

$$H_{a^*} \leq_1 H_A(|a|) \leq_1 H_{a^{**}}.$$

Now, with the aid of Theorem 7, from Theorem 6 we obtain

Corollary (Kleene (4, 5)). A set X ($\subseteq N^{k+1}$) belongs to

$$\Sigma_1^1 \cap \Pi_1^1$$

if and only if X is hyperrecursive relative to H_a for some $a \in O$.

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Note: Figure translations are in progress. See original paper for figures.

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