

THE BIRTH OF PERIODIC MOTIONS FROM A STATE OF EQUILIBRIUM

G. Ya. Shafranov

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Abstract

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MATHEMATICS

G. Ya. SHAFRANOV

THE BIRTH OF PERIODIC MOTIONS FROM A STATE OF EQUILIBRIUM

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In the present note we consider conditions for the appearance of limit cycles from the system

$$\dot{x} = X(x), \quad (\text{A})$$

which has an equilibrium state with two purely imaginary roots (the remaining ones having nonzero real parts) and a Lyapunov quantity $g_{2m+1} \neq 0$, in passing to a δ -close, up to rank $2m + 4$ (see ⁽¹⁾), system

$$\dot{x} = X(x) = P(x) = \tilde{X}(x). \quad (\tilde{A})$$

A relation is established between the number of generated limit cycles and the order of the Lyapunov quantity g_{2m+1} ; a complete classification of the trajectories of system $(\tilde{A})$ is given. Here x is a vector; $X(x), P(x)$ are vector-functions belonging to the class C^N ($N \geq 2m + 4$) (to the analytic class).

In ⁽¹⁾ conditions for the appearance of limit cycles from an equilibrium state of a two-dimensional system were studied; in ^(2, 3), conditions for the birth of one periodic motion from a system having two purely imaginary roots and n negative roots, $g_3 \neq 0$. The general case has not hitherto been considered by anyone. The study is carried out by the method of point mappings; for its construction the system is reduced to a special form, and the neighborhood is split into a noncritical region, in which there are no closed trajectories, and a critical one.

§ 1. **Lemma 1.** *The system $(\tilde{A})$, by a nondegenerate change of variables and time, can be reduced to the form*

$$\begin{aligned}
\dot{x} &= -y + g_{2m+1}x(x^2 + y^2)^m + X(x, y, z, w) + P(x, y, z, w), \\
\dot{y} &= x + g_{2m+1}y(x^2 + y^2)^m + Y(x, y, z, w) + Q(x, y, z, w), \\
\dot{z} &= A^-z + Z(x, y, z, w) + L(x, y, z, w), \\
\dot{w} &= A^+w + W(x, y, z, w) + M(x, y, z, w),
\end{aligned} \tag{1}$$

where x, y are scalars; z, w are vectors; A^- , A^+ are matrices whose characteristic roots have negative and, respectively, positive real part. X, Y, Z, W, P, Q, L, M belong to the class C^{N-1} (to the analytic class); X, Y, W vanish for $x = 0$, $y = 0$, $w = 0$; $X(x, y, 0, 0)$, $Y(x, y, 0, 0)$, $Z(x, y, 0, 0)$, $W(x, y, 0, 0)$ vanish for $x = y = 0$ together with their derivatives up to order $2m+1$; X, Y, Z, W are obtained from the right-hand sides of system (A), while P, Q, L, M are obtained from the additions.

For the proof, the ideas of the monographs ^(4, 5) are used.

§ 2. Lemma 2 (on an invariant surface). Suppose that in some neighborhood $G_x \times G_y$ of the fixed point $x = 0$, $y = 0$, where x, y are vectors, a point mapping is given

$$\bar{x} = Ax + X(x, y), \quad \bar{y} = By + Y(x, y), \tag{2}$$

where the spectral radius of the constant matrix B , $\|B\|_{\text{sp}} = r_2$, matrices

$$A^{-1}\|A^{-1}\|_{\text{sp}}^{-1} = r_1, \quad r_1 > r_2, \quad r_1 > 1; \quad X(x, y), \quad Y(x, y)$$

are vector functions that vanish, together with their first-order derivatives, at $x = 0$, $y = 0$, and belong to the class C^k , $k \geq 1$.

Then, in some neighborhood of the origin, there exists an invariant surface $y = \varphi(x)$ belonging to the class C^{k-1} , $\varphi(0) = 0$ ($\varphi'_x(0) = 0$ for $k > 1$), satisfying the Poincaré functional equation

$$\varphi[Ax + X(x, \varphi(x))] = B\varphi(x) + Y(x, \varphi(x)). \tag{3}$$

Lemma 3. Suppose that $\det A \neq 0$, $\|A\|_{\text{sp}} = r_1$, $\|B^{-1}\|_{\text{sp}}^{-1} = r_2$, $r_1 < r_2$, $r_1 < 1$.

Then the mapping (2) has an invariant surface $x = \psi(y)$ (uniqueness may fail), $\psi(0) = 0$, $\psi(y) \in C^{N_0-1}$, where $1 \leq N_0 \leq k$.

Lemmas 2 and 3 are proved by constructing a metric space of surfaces and then applying the contraction mapping principle.* On the basis of Lemma 3, one establishes the existence of smooth invariant surfaces $E_1 : w = \psi(r, z)$, $\psi(0, 0) = 0$, and $E_2 : z = \chi(r, w)$, $\chi(0, 0) = 0$, of dimensions $p+1$ and $q+1$, respectively, defined in some neighborhood of the fixed point of the point mapping

$$\begin{aligned}
\bar{r} &= r_0 + f_1(r_0, z_0, w_0), \\
\bar{z} &= s^-z_0 + f_2(r_0, z_0, w_0), \\
\bar{w} &= s^+w_0 + f_3(r_0, z_0, w_0),
\end{aligned} \tag{T}$$

where $r_0, \bar{r}$ are scalars; $z_0, \bar{z}$ are p -dimensional vectors; $w_0, \bar{w}$ are q -dimensional vectors; s^- is a constant $(p \times p)$ matrix with spectrum inside the unit circle, $\det s^- \neq 0$; s^+ is a constant $(q \times q)$ matrix with spectrum outside the unit circle; f_1, f_2, f_3 are nonlinearities belonging to the smoothness class C^{N-1} . Considering the mapping (T) on one of the surfaces of the form $w = \psi(r, z)$ and then selecting the invariant surface $z = \psi_1(r)$, $\psi_1(0) = 0$, we obtain

$$\bar{r} = r_0 + (G_k + f_4(r_0))r_0^k, \quad k = 2, \dots, N-1, \quad (4)$$

where $f_4(0) = 0$, and suppose $G_k \neq 0$ is a quantity which we shall call Lyapunov. An invariant manifold containing the fixed point O will be called incoming (outgoing) if T^{nM} tends asymptotically to O as $n \rightarrow \infty$ ($n \rightarrow -\infty$). Denote by $C_{i,j}^1$, $i+j = p+q+1$, a saddle fixed point with an i -dimensional incoming and a j -dimensional outgoing manifold. Then the following holds.

Lemma 4. 1) Suppose $G_k > 0$, k odd; then we have a saddle $C_{p,q+1}^1$; 2) $G_k < 0$, k odd; we have a saddle $C_{p+1,q}^1$; 3) k even; then the fixed point has a $(p+1)$ -dimensional incoming manifold with boundary and a $(q+1)$ -dimensional outgoing manifold with boundary.

We shall denote such a saddle point by $C_{p+1/2,q+1/2}^1$, and in what follows we shall regard it as the merging of the saddles $C_{p+1,q}^1$ and $C_{p,q+1}^1$. Trajectories (T^{nM} for increasing and decreasing n) that do not belong to the incoming and outgoing invariant manifolds will be saddle trajectories (cf. ⁽¹¹⁾).

To construct the point mapping in the critical region

$$|w|^2 \leq \sigma r^2, \quad \text{where } r^2 = x^2 + y^2, \quad \sigma > 0, \quad |w|^2 = w_1^2 + \dots + w_q^2, \quad (5)$$

where w_i are the components of the vector w , in equations (1) we make the substitution

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z, \quad w = r \bar{w} \quad (6)$$

in the region (5), $|w|^2 \leq \sigma r^2$, which leads to regular expressions.

* Invariant surfaces of point mappings were studied in ⁽⁶⁻⁸⁾. On invariant surfaces of differential equations, see ⁽¹⁰⁾.

The mapping of the surface $\varphi = 0$ onto the surface $\varphi = 2\pi$ will take the form

$$\begin{aligned} \bar{r} &= sr_0 + f_1(r_0, z_0, w_0), \\ \bar{z} &= s^- z_0 + f_2(r_0, z_0, w_0), \\ \bar{w} &= s^+ w_0 + f_3(r_0, z_0, w_0). \end{aligned} \quad (7)$$

Here, when the additions are absent, $s = 1$; $s^- = \exp(2\pi A^-)$ is a $(p \times p)$ -matrix, $s^+ = \exp(2\pi A^+)$ is a q -dimensional matrix whose characteristic roots for s^- lie inside, and for s^+ outside, the unit circle; $\det s^- \neq 0$. The fixed points of the

mapping (7) (with the exception of the trivial one $r_0 = 0$, $z_0 = 0$, $w_0 = 0$, corresponding to the equilibrium state) correspond to limit cycles of equation (1). Reducing the problem on fixed points of the mapping (7) to the problem of determining the roots r_0^* of a defining scalar equation, we see that there will be no more than m positive roots if $g_{2m+1} \neq 0$ is a Lyapunov quantity.

Lemma 5. *In some neighborhood, independent of the particular choice of a system $(\tilde{A})$ that is δ -close up to order $2m + 4$ to the system (A) , and of the point mapping (T) , there exists an invariant surface of the form $w = \psi(r, z)$, passing through all fixed points of the mapping.*

Denote by $C_{i,j}$ a cycle in the $(p + q + 2)$ -dimensional space, $p + q + 2 = i + j - 1$, which is the intersection of the i -dimensional incoming and the j -dimensional outgoing manifolds. Under the point mapping, the cycle C_{ij} corresponds to a saddle fixed point $C_{i-1,j-1}^1$. Restricting ourselves to cycles of integer multiplicity, we obtain the main result.

Theorem 1. *Let a system (A) of class C^N (of analytic class), with two purely imaginary roots, p roots with negative real part, and q roots with positive real part, have the Lyapunov quantity $g_{2m+1} \neq 0$ ($2m + 4 \leq N$).*

Then for a system $(\tilde{A})$ of class C^N (of analytic class) the following holds: a) there exist $\varepsilon_0 > 0$, $\delta_0 > 0$ such that, whatever system $(\tilde{A})$, δ -close up to order $2m + 4$, we take, in the ε_0 -neighborhood of O (the equilibrium state) there will be no more than m closed trajectories; b) for any $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ one can specify a system $(\tilde{A})$ for which, in the ε -neighborhood of the point O , there will lie a prescribed number k ($0 \leq k \leq m$) of limit cycles; c) the cycles $C_{p+2,q+1}$ and $C_{p+1,q+2}$ will alternate (will be adjacent in r_0); adjacent to the equilibrium state $O_{p+2,q}$ will be the cycle $C_{p+1,q+2}$, and adjacent to $O_{p,q+2}$ will be the cycle $C_{p+2,q+1}$; d) a two-dimensional manifold whose trajectories, as $t \rightarrow \infty$, tend to one limit cycle $C_{p+2,q+1}(O_{p+2,q})$, and as $t \rightarrow -\infty$ tend to the cycle $C_{p+1,q+2}(O_{p,q+2})$, is the intersection of the incoming $(p + 2)$ -dimensional manifold of one cycle and the $(q + 2)$ -dimensional manifold of the other cycle (or equilibrium state); e) all other trajectories not belonging to invariant incoming and outgoing manifolds pass at a finite distance from the generated cycles (and the equilibrium state).

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Scientific Research Institute
of Applied Mathematics and Cybernetics
at Gorky State University
named after N. I. Lobachevsky

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