

NONDISCRETE TOPOLOGIZABILITY OF INFINITE COMMUTATIVE RINGS

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Abstract**Full Text**

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MATHEMATICS

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NONDISCRETE TOPOLOGIZABILITY OF INFINITE COMMUTATIVE RINGS*(Presented by Academician P. S. Aleksandrov on 17 III 1970)*

It is obvious that in any ring one can introduce a topology in a trivial way (the discrete topology). In finite rings this is the only possible topology. The question of the possibility of introducing a nondiscrete topology in certain infinite rings was considered in papers ^(1,3,4,6). In the present paper it is proved that every infinite commutative ring admits a nondiscrete topologization.

Theorem 1. If a commutative ring R has an infinite nil-ideal, then R has an infinite ideal I such that $I^2 = 0$.

Lemma 1. If a primary group G for the prime number p contains only a finite number of elements of order p , then G is a finite direct sum of finite cyclic groups and groups of type p^∞ .

Lemma 2. If a commutative ring R has an infinite ideal I such that $I^2 = 0$ and $p^k I = 0$ for some power k of the prime number p , then R admits a nondiscrete topologization in which I is an open ideal.

Proof. By the first theorem of Prüfer (see ⁽⁵⁾, p. 146), I as a group decomposes into a direct sum of cyclic subgroups G_α , i.e.

$$I = \sum_{\alpha \in \Omega} G_\alpha.$$

If S is a finite subset of Ω , then put

$$N_S = \sum_{\alpha \in S} G_\alpha.$$

For any finite subsets $S \subset \Omega$ and $M \subset R$, define the set

$$V_{M,S} = \{a \in N_S \mid aM \subseteq N_S\}.$$

It is easy to verify that the family of subsets $V_{M,S}$ can be taken as a base of neighborhoods of zero, in order to turn R into a nondiscrete topological ring.

Lemma 3. If a commutative ring R has an infinite ideal I such that $I^2 = 0$ and I , as a group, is a direct sum of a finite number of groups of type p^∞ for some prime number p , then R admits a nondiscrete topologization in which I is an open ideal.

Proof. Let

$$I = \sum_{i=1}^n G_i,$$

where G_i are groups of type p^∞ . Then there exist elements $b_{ij} \in G_j$, $j = 1, 2, \dots; i = 1, 2, \dots, n$, such that

$$pb_{i,j+1} = b_{ij}$$

and

$$pb_{i1} = 0.$$

It is obvious that any element $a \in I$ can be written uniquely in the form of a finite sum

$$\sum_{i,j} a_{ij} b_{ij},$$

where a_{ij} are integers, and moreover $|a_{ij}| \leq p/2$. To the element

$$a = \sum_{i,j} a_{ij} b_{ij}$$

we assign the number

$$\xi(a) = \sum_{i,j} \frac{|a_{ij}|}{p^j}.$$

For each finite subset $A \subset R$ and any natural number m , define the set

$$V_{A,m} = \{a \in I \mid \xi(a) \leq 1/2^m \text{ and } \xi(ab) \leq 1/2^m \text{ for any } b \in A\}.$$

It is proved that the collection $\{V_{A,m} \mid A \subset R, m = 1, 2, \dots\}$ can be taken as a base of neighborhoods of zero, so as to turn R into a nondiscrete topological ring.

Recall that a subgroup H is called densely embedded in a group G if H has nonzero intersection with every nonzero subgroup of the group G .

Remark. It is obvious that every nonzero torsion-free group G contains an infinite densely embedded reduced subgroup H .

Lemma 4. *If a commutative ring R possesses a nonzero ideal I such that $I^2 = 0$ and I is a torsion-free group, then R admits a nondiscrete topologization in which I is an open ideal.*

Proof. Let H be a densely embedded reduced subgroup in the additive group of the ideal I . For any finite subset $M \subset R$ and any natural number n , define the set

$$V_{M,n} = \{a \in nH \mid aM \subset nH\}.$$

It is easy to verify that the collection $\{V_{M,n} \mid M \subset R, n = 1, 2, \dots\}$ can be taken as a base of neighborhoods of zero, so as to turn R into a nondiscrete topological ring.

Lemma 5. *If I is an ideal of the ring R and B is the largest complete subgroup of the additive group of the ideal I , then B is an ideal in R .*

Lemma 6. *Let I be an ideal of the ring R , and let A be the set of all elements of finite order in I . If A has finite characteristic and R/A admits a nondiscrete topologization in which I/A is an open ideal, then R also admits a nondiscrete topologization in which I is an open ideal.*

Proof. Let n be the characteristic of the ring A , and let $\{\overline{V}_\alpha \mid \alpha \in \Omega\}$ be a base of neighborhoods of zero in the ring R/A . If φ is the canonical homomorphism of the ring R onto R/A , then put

$$U_\alpha = n\varphi^{-1}(\overline{V}_\alpha).$$

It is obvious that the collection of sets $U_\alpha, \alpha \in \Omega$, can be taken as a base of neighborhoods of zero, so as to turn R into a nondiscrete topological ring, with I an open ideal.

Theorem 2. *Every commutative ring R containing an infinite nil ideal I admits a nondiscrete topologization in which I is an open ideal.*

Proof. By Theorem 1, R has an infinite ideal I_0 such that $I_0^2 = 0$. If A is the set of all elements of I_0 having finite order, then A is an ideal in R .

Consider two cases:

I. A is a finite ideal. Then $nA = 0$ for some number n . Since $\overline{R} = R/A$ contains an infinite ideal $\overline{I}_0 = I_0/A$ such that $\overline{I}_0^2 = 0$ and $\overline{I}_0$ contains no elements of finite order, it follows from Lemma 4 that $\overline{R}$ admits a nondiscrete

topologization in which $\bar{I}_0$ is an open ideal. Then, by Lemma 6, R admits a nondiscrete topologization in which I_0 is an open ideal.

- II. A is an infinite ideal. Let $\sum_i C_i$ be a decomposition of A into a direct sum of ideals C_i , the additive group of each of which is primary for some prime number p_i . If the sum $\sum_i C_i$ contains infinitely many summands, then the collection of ideals

$$V_n = \sum_{i=n}^{\infty} C_i$$

defines some nondiscrete topology in R . Suppose

$$A = \sum_{i=1}^r C_i,$$

then some C_{i_0} is infinite. If C_{i_0} contains infinitely many elements of order p_{i_0} , then, by Lemma 2, R admits a nondiscrete topologization in which I is open-

ideal. If, however, C_{i_0} contains only a finite number of elements of order p_{i_0} , then from Lemmas 1 and 5 it follows that R has such a nonzero ideal B that $B \subseteq I_0$ and the additive group of the ideal B is a finite direct sum of groups of type p^∞ . By Lemma 3, R admits a nondiscrete topology in which I is an open ideal. This proves the theorem.

Lemma 7. *If a commutative ring R has an infinite ideal without nilpotent elements, then R admits a nondiscrete topology.*

Proof. If R has such an infinite ideal I that I contains no minimal ideals of the ring R , then there exists a decreasing transfinite sequence of nonzero ideals $I_1 \supset I_2 \supset \dots$ in R such that $\bigcap_{\alpha} I_{\alpha} = 0$. Obviously, the collection of ideals I_{α} defines a nondiscrete topology in R .

Suppose now that every ideal of the ring R contains a minimal ideal of the ring R , and let I_1 be some infinite ideal without nilpotent elements in R . If I_1 satisfies the minimality condition for ideals, then I_1 is a finite direct sum of fields A_i . Since I_1 is an infinite ideal, some A_{i_0} is an infinite field. According to (6), a nondiscrete topology can be defined in A_{i_0} . From the fact that $R = A_{i_0} \oplus B$, it follows that a nondiscrete topology can also be defined in R . If, however, I_1 does not satisfy the minimality condition, then there exists an infinite number of fields $A_1, A_2, \dots$ such that $A_i \subset I_1$ and the A_i are ideals in R . Then the collection of ideals

$$V_n = \sum_{i=n}^{\infty} A_i$$

defines a certain nondiscrete topology in R .

Lemma 8. *If an infinite commutative ring R of finite characteristic contains only a finite number of nilpotent elements and every nonzero ideal contains a nonzero nilpotent element, then R admits a nondiscrete topology.*

Proof. Without loss of generality, we may assume that the characteristic of the ring R is a power of a prime number p . For any subset $S \subseteq R$, by S^* we shall denote the annihilator of the set S in R .

If I is the set of all nilpotent elements of the ring R , then I is a finite ideal and, consequently, $I^{p^k} = 0$ for some number k . Consider the set $M = \{a \in R \mid \{a\}^* \cap C \not\subseteq I \text{ for any nonnilpotent ideal } C \text{ of the ring } R, \text{ contained in } I^*\}$. For any finite subset $Q \subset M$, define the set

$$V_Q = \left\{ \sum d_i^{p^k} \mid d_i \in Q^*, \, p d_i = 0 \right\}.$$

It can be verified that the collection of sets V_Q , $Q \subset M$, defines a certain nondiscrete topology in R .

Lemma 9. *Every infinite commutative ring R of finite characteristic, containing only a finite number of nilpotent elements, admits a nondiscrete topology.*

Proof. Let $\mathfrak{N}$ be the set of all ideals in R that do not contain nilpotent elements, and let A be some maximal ideal in $\mathfrak{N}$. If A is an infinite ideal, then, by Lemma 7, R admits a nondiscrete topology. If, however, A is a finite ideal, then A has an identity element. Then $R = A \oplus B$. Since A is a maximal ideal in $\mathfrak{N}$, every ideal of the ring B contains a nonzero nilpotent element. By Lemma 8, B admits a nondiscrete topology. Then a nondiscrete topology can also be defined in R .

Theorem 3. *Every infinite commutative ring R admits a nondiscrete topology.*

Proof. If R contains an infinite number of nilpotent elements, then, by Theorem 2, R admits a nondiscrete topology.

Let now R contain only a finite number of nilpotent elements, and let A be the set of all elements of finite order of the ring R . Consider two cases:

I. A has finite characteristic. If $A = R$, then, by Lemma 9, R admits a nondiscrete topologization. If $A \neq R$, then from Theorem 2 and Lemmas 7 and 8 it follows that R admits a nondiscrete topologization.

II. A does not have finite characteristic. Let $\sum_i G_i$ be the decomposition of the additive group of the ideal A into primary subgroups G_i with respect to the prime numbers p_i . Then the G_i are ideals in R . From the fact that R contains only a finite number of nilpotent elements it follows that each G_i has finite characteristic. Hence $\sum G_i$ contains an infinite number of summands. It is clear that the collection of ideals

$$V_n = \sum_{i=n}^{\infty} G_i$$

defines a certain nondiscrete topology in R . This completes the proof of the theorem.

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