

CLASSES OF CONJUGATE SUBGROUPS

V. I. Ushakov

1970

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-197001.91045>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.46

MATHEMATICS

V. I. USHAKOV

CLASSES OF CONJUGATE SUBGROUPS IN TOPOLOGICAL GROUPS

(Presented by Academician P. S. Novikov on 26 V 1969)

Let G be an arbitrary topological group, and H its closed subgroup. Denote by S_H the class of subgroups of the group G conjugate to H . On the set S_H , whose elements are the subgroups $g^{-1}Hg$, one can introduce a topology by declaring a neighborhood of the element $g^{-1}Hg$ to be the collection of subgroups of the form $u^{-1}Hu$, where u runs through some neighborhood U of the element g . We obtain a topological space which, as is easy to show, is homeomorphic to the quotient space G/N of the group G by the normalizer $N = N(H)$ of the subgroup H (the normalizer of a closed subgroup is closed—see ⁽³⁾), and the homeomorphism is given as follows:

$$x^{-1}Hx \longleftrightarrow Nx.$$

In the author's paper ⁽²⁾, groups were studied for which all classes of conjugate subgroups are bicomact. In the case of discrete groups this condition becomes the condition that the classes of conjugate subgroups be finite. Such groups were considered by Neumann ⁽¹⁾.

In the present paper we shall consider groups in which only the classes of conjugate **bicomact** subgroups are bicomact. Some results of ⁽²⁾ admit strengthening, as is shown by the

Theorem. 1) *If G is a locally bicomact group in which all classes of conjugate bicomact subgroups with one topological generator are bicomact, then the set P of all bicomact elements of the group G forms a closed locally normal subgroup (the periodic part of G), the quotient group by which is pure, i.e. contains no bicomact elements distinct from the identity.*

2) *If all classes of conjugate bicomact subgroups of a locally bicomact group G are bicomact, then G is projectively Lie, i.e. in every neighborhood of its identity there is a bicomact normal divisor such that the quotient group by it is a Lie group.*

- 3) If in a locally bicomact group G the classes of conjugate subgroups with one topological generator are bicomact, then G is an $\overline{FS}$ -group, i.e. the closures of all its classes of conjugate elements are bicomact.

Recall that a bicomact element is an element contained in a bicomact subgroup, while a locally normal group is a group each element of which lies in a bicomact normal divisor.

For the proof of the theorem we shall need the

Lemma. Let H be a bicomact subgroup of a locally bicomact group G , and let the space S_H be bicomact. Then H is contained in a bicomact normal divisor of the group G .

Proof. Let H be bicomact and let N be the normalizer of H , $\tilde{G} = G/N$. By virtue of the homeomorphism of S_H and $\tilde{G}$, the quotient space $\tilde{G}$ is bicomact. Take in G a neighborhood U of the identity with bicomact closure $\overline{U}$. By $\overline{U}g$ denote the set Nug of right cosets, where u runs through the neighborhood U . From the covering of the bicomact set $\tilde{G}$

by the domains $\tilde{U}g$, choose a finite subcovering $\tilde{U}g_1, \dots, \tilde{U}g_k$. Then the group G will be the set-theoretic sum of its subsets $NUg_1, \dots, NUg_k$. Let g be an arbitrary element of G . It has the form xug_i , where $x \in N$, $u \in U$, $1 \leq i \leq k$. Then

$$g^{-1}Hg = g_i^{-1}u^{-1}x^{-1}Hxug_i = g_i^{-1}u^{-1}Hug_i.$$

It follows that the set $B = \bigcup g^{-1}Hg$ is contained in the bicomact set

$$C = \bigcup_{i=1}^k g_i^{-1}\overline{U}^{-1}H\overline{U}g_i$$

and therefore has bicomact closure. Further, B is invariant and consists of bicomact elements. Finally, in [4] (see Theorem 3) it was shown that every invariant set B of a locally bicomact group G , consisting of bicomact elements and having bicomact closure, generates, in the topological sense, the bicomact normal divisor $\overline{\{B\}}$ of the group G (in Theorem 3 of [4] it was required that the set B itself be bicomact, but this requirement can be weakened, and no changes in the proof are needed). The inclusion $H \subset \overline{\{B\}}$ proves the lemma.

We pass to the proof of the first assertion of the theorem. Let a_1 and a_2 be two bicomact elements of the group G , $A_1 = \{a_1\}$, $A_2 = \{a_2\}$. By the lemma, A_1 and A_2 are contained in bicomact normal divisors N_1 and, respectively, N_2 of the group G . Then the element a_1a_2 is contained in the bicomact normal divisor N_1N_2 , whence it follows that the set P is a subgroup in the algebraic sense. The closedness of P and the purity of G/P follow from (5).

Let us prove the second assertion of the theorem. It is known (see [6], Theorem 9) that a locally bicomact group G contains an open projective-left subgroup H . Choose, in an arbitrary neighborhood $U \subset H$ of the identity, a bicomact

normal divisor B of the group H such that the quotient group H/B is a Lie group. The normalizer N of the subgroup B contains H , and therefore is open in G and, defining a bicomact quotient space, has finite index in G . The intersection N' of all subgroups conjugate to N will be an open normal divisor of the group G of finite index. The subgroup $B \cap N'$ is invariant in N' .

Consider the quotient group $H \cap N' / B \cap N'$. It is locally bicomact and admits a continuous one-to-one homomorphism into the Lie group H/B :

$$H \cap N' / B \cap N' = H \cap N' / B \cap (H \cap N') \cong (H \cap N') B / B \subset H/B.$$

By Cartan's theorem (see [7], pp. 190–198), $H \cap N' / B \cap N'$ is a Lie group, and since the subgroup $H \cap N'$ is open in N' , it follows that $N' / B \cap N'$ is also a Lie group. We thus have an invariant open subgroup N' in G and a bicomact normal divisor $B' = B \cap N'$ of the group N' , lying in the neighborhood U and defining the Lie quotient group N' / B' . The normalizer of the subgroup B' contains N' and, consequently, has finite index. This means that B' has a finite number of subgroups conjugate to it,

$$g_1^{-1} B' g_1, \dots, g_n^{-1} B' g_n.$$

From the invariance of N' in G it follows that the transformation of N' by elements of G is its topological automorphism. Hence all the subgroups $g_i^{-1} B' g_i$, $i = 1, \dots, n$, are invariant in N' and define Lie quotient groups. The intersection B'' of these subgroups will be a normal divisor in G . The quotient group N' / B'' is a Lie group by Lemma 3 of [6]. Since G / N' is a discrete group, G / B'' is a Lie group.

We shall prove the last assertion of the theorem. If g is a bicomact element of the group G , i.e., the subgroup $\{g\}$ is bicomact, then, by the lemma, $\overline{\{g\}}$ is contained in a bicomact normal divisor of the group G , whence it follows that g is an FC -element. If the element g is not bicomact, then it is pure, i.e., it generates a discrete (and therefore closed) infinite cyclic subgroup $\{g\}$. By hypothesis, the factor space G/N , where N is the normalizer of $\{g\}$, is bicomact. Transformation by means of elements of N induces in $\{g\}$ some automorphism, and under an automorphism the property of an element of being a generator is preserved. Since in $\{g\}$ there exist only two generating elements, g and g^{-1} , g has in N at most two conjugate elements, whence it follows that $N/Z_N(g)$ is a finite factor space (by $Z_N(g)$ we denote the centralizer of $\{g\}$ in N). Since, further, $Z_N(g) = Z_G(g)$, the factor space $G/Z_G(g)$ is bicomact. In that case the class of elements conjugate to g , homeomorphic to this factor space, is also bicomact. Thus the pure elements have turned out to be even FC -elements, i.e., to determine bicomact classes of conjugate elements. Analogous arguments show that all elements of finite order are also FC -elements. The theorem is completely proved.

From the first assertion of the theorem we obtain

Corollary. *A periodic* locally bicomact group in which the bicomact classes of conjugate subgroups topologically generated by one element are bicomact is locally normal.*

In the case of discrete groups this assertion admits a converse: in a locally normal discrete group all classes of conjugate finite subgroups are finite. In the general case such a converse is impossible, as is shown by the example, cited in ⁽⁸⁾, of a topological wreath product of a cyclic group of order 2 and a group of type 2^∞ .

Moscow State Pedagogical Institute
named after V. I. Lenin

Received
11 V 1969

CITED LITERATURE

- ¹ B. H. Neumann, Math. Zs., **63**, No. 1, 76 (1955).
- ² V. I. Ushakov, Matem. sborn., **63**, 2, 277 (1964).
- ³ A. G. Kurosh, Izv. AN SSSR, ser. matem., **9**, 65 (1945).
- ⁴ V. I. Ushakov, Sibirsk. matem. zhurn., **4**, No. 5, 1162 (1963).
- ⁵ V. P. Platonov, ibid., **7**, No. 4, 854 (1966).
- ⁶ V. M. Glushkov, UMN, **12**, issue 2 (74), 3 (1957).
- ⁷ C. Chevalley, *Theory of Lie Groups*, 1, IL, 1958.
- ⁸ V. I. Ushakov, Sibirsk. matem. zhurn., **4**, No. 3, 689 (1963).

* A topological group is called periodic if all its elements are bicomact.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.