

On the Spectrum of Acoustic Turbulence

V. E. Zakharov, R. Z. Sagdeev

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Abstract

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UDC

HYDROMECHANICS

V. E. ZAKHAROV, Academician R. Z. SAGDEEV

ON THE SPECTRUM OF ACOUSTIC TURBULENCE

1. Acoustic turbulence is understood as turbulence of a compressible fluid in which the flow of the fluid is potential and consists of an ensemble of interacting sound waves (sound noise).

Let us introduce normal variables—the complex amplitudes of traveling waves—by the formulas

$$\rho_k = \frac{|k|}{\sqrt{2\omega_k^{1/2}}} \rho_0^{1/2} (a_k + a_{-k}^*), \quad \omega_k = c(k),$$

$$\Phi_k = -\frac{i\omega_k^{1/2}}{\sqrt{2}|k|\rho_0^{1/2}} (a_k - a_{-k}^*). \quad (1)$$

Here c is the speed of sound; ρ_k and Φ_k are the Fourier transforms of the density and the hydrodynamic potential. In these variables the hydrodynamic equations have the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i \int V_{kk_1 k_2} (a_{k_1} a_{k_2} \delta_{k-k_1-k_2} + 2a_{k_1}^* a_{k_2} \delta_{k+k_1-k_2} + a_{k_1}^* a_{k_2}^* \delta_{k+k_1+k_2}) dk_1 dk_2; \quad (2)$$

$$V_{kk_1 k_2} = \frac{1}{16} \left(\frac{c}{\pi^3 \rho_0} \right)^{1/2} \left\{ \frac{|k|^{1/2}}{|k_1| |k_2|^{1/2}} (k_1 k_2) + \frac{|k_1|^{1/2}}{(|k| |k_2|)^{1/2}} (k k_2) + \right.$$

$$\left. + \frac{|k_2|^{1/2}}{(|k| |k_1|)^{1/2}} (k k_1) + 3q (|k| |k_1| |k_2|)^{1/2} \right\}. \quad (3)$$

Here it is assumed that the wave amplitudes are small ($\delta\rho/\rho \ll 1$), and one may use the expansion of the internal energy of the fluid in powers of the density variation

$$\varepsilon(\rho) = \varepsilon_0 + \frac{1}{2} \frac{c^2}{\rho_0} \left(\delta\rho^2 + q \frac{\delta\rho^3}{\rho_0} \right).$$

In this case, to within higher-order terms, the spectral energy density has the form

$$\varepsilon_k = 4\pi k^2 \omega_k \langle |a_k|^2 \rangle. \quad (4)$$

In the lowest order of perturbation theory, sound waves interact while obeying the conservation laws

$$\omega_k = \omega_{k_1} + \omega_{k_2}, \quad k = k_1 + k_2.$$

These laws can be satisfied only for collinear vectors directed in the same direction. As for processes of higher orders, their amplitudes have strong singularities when the wave vectors of the interacting waves are directed along one straight line. Therefore one may assume that a wave having wave vector k_0 interacts only with those waves whose wave vectors lie in a very narrow cone with axis k_0 . To estimate the width of this cone, let us pass to a reference frame moving in the direction k_0 with velocity c and expand $|k|$ near k_0 . Linearly—

the term in equation (2) will take the form

$$ic(|\mathbf{k}| - |\mathbf{k}_0|)a_k \simeq ic \left(k_z + \frac{1}{2} \frac{k_\perp^2}{k_0} \right) a_k.$$

Next, by the change of variable $a_k = a_k^* e^{-ick_z t}$, we eliminate the term $ick_z a_k$. We now determine the quantity $k_\perp^{02}$ by comparing, in order of magnitude, the linear and quadratic terms in equation (2):

$$c \frac{k_\perp^{02}}{k_0} a^* \sim V a^{*2} k_0^3.$$

Hence

$$k_\perp^{02} \sim \frac{V}{c} a^* k_0^4 \sim k_\parallel^2 \frac{\delta\rho}{\rho}. \quad (5)$$

The quantity $k_\perp^0$ obviously represents the characteristic transverse size of the interaction cone, since for $k_\perp \gg k_\perp^0$ the interaction of waves becomes nonresonant and strongly weakens.

The characteristic time of nonlinear interaction can be estimated by the formula

$$\frac{1}{\tau} \sim V a^* k_0 k_{\perp}^2 \sim \frac{V^2}{c} a^{*2} k_0^5. \quad (6)$$

Let us also note that, because of the specific character of the interaction of sound waves, in formula (3) one may approximately replace scalar products by products of moduli; this gives

$$V_{kk_1k_2} \simeq 3/16 (c/\pi^3 \rho_0)^{1/2} (1+q)(|\mathbf{k}| |\mathbf{k}_1| |\mathbf{k}_2|)^{1/2}. \quad (7)$$

The singularities of the amplitudes of higher-order processes do not allow one, even for waves of small amplitude, to restrict oneself to low order and pass to the theory of weak coupling. Such a transition is possible only in the presence of sound dispersion, when $\omega_k = ck(1 + \varepsilon k^2)$; in this case the condition $\rho^{-1} \delta \rho \ll \varepsilon k^2$ must be satisfied. Therefore, for the case of strong coupling ($\varepsilon = 0$) we shall use dimensional estimates. Adopting the Kolmogorov hypothesis (^{2, 3}), we shall assume that the turbulence is locally isotropic and that its spectrum is completely determined by the single quantity—the energy flux into the region of large k , which may be defined by the formula

$$P = \frac{\partial}{\partial t} \int_0^\infty \varepsilon_k dk \simeq \frac{\varepsilon_k k}{\tau}, \quad (8)$$

where τ is the characteristic time of nonlinear interaction.

Because, in comparison with an incompressible fluid, there is an additional quantity—the speed of sound c —dimensional considerations are insufficient to determine the spectrum; from them it follows only that the spectrum has the form

$$\varepsilon_k = \frac{\rho_0 c^2}{k} f\left(\frac{P}{\rho_0 c^3 k}\right), \quad (9)$$

where f is as yet an unknown function.

To determine f , it is necessary to know the dependence of the interaction time τ on the spectral energy function. For an incompressible fluid this relation is determined from the estimate $\tau^{-1} \sim kV \sim \varepsilon^{1/2}$, which gives $f(\xi) = \xi^{2/3}$, $\varepsilon_k \sim k^{-5/3}$ —the only case in which the speed of sound disappears from formula (9).

In the case under consideration the interaction time is determined by formula (6), and the energy spectrum by formula (4); from these formulas it follows that $1/\tau \sim \varepsilon$. Hence we have

$$P \sim \varepsilon^2, \quad f(\xi) = \xi^{1/2}, \quad \varepsilon_k \simeq \rho_0^{1/2} c^{1/2} P^{1/2} / k^{3/2}. \quad (10)$$

As for an incompressible fluid, the integral $\int_0^\infty \varepsilon_k dk$ diverges in the region of small k . Accordingly, one can divide k -space into energy-containing, inertial, and dissipative regions; spectrum (10) is realized in the inertial region.

2. Let us now compare the result obtained with what is given by the theory of weak turbulence. In this theory the quantity $n_k = \langle |a_k|^2 \rangle$ obeys the kinetic equation

$$\begin{aligned} \frac{\partial n_k}{\partial t} = 2\pi \int |V_{kk_1/2}|^2 \{ (n \cdot n_{k_2} - n_k n_{k_1} - n_k n_{k_2}) \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \delta_{\omega_k-\omega_{k_1}-\omega_{k_2}} + \\ + 2(n_{k_1} n_{k_2} + n_k n_{k_2} - n_k n_{k_1}) \delta_{\mathbf{k}1-\mathbf{k}-\mathbf{k}_2} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}} \} d\mathbf{k}_1 d\mathbf{k}_2, \quad \varepsilon_k = 4\pi k^2 \omega_k n_k. \end{aligned} \quad (11)$$

Let us perform averaging over the angles in k -space. As a result, the δ -function of the wave numbers will be replaced by the factor $f(k, k_1, k_2) = 2\pi/kk_1k_2$.

It is clear that in equation (11) one may, without fear of contradictions, neglect the dispersion and take $\omega_k = ck$; $\varepsilon_k = 4\pi ck^3 n_k$. After these simplifications, equation (11) can be transformed to the form

$$\partial \varepsilon_k / \partial t + \partial P_k / \partial k = 0; \quad (12)$$

$$\begin{aligned} P_k = -16\pi^3 \int_0^k k' dk' \int_{k-k'}^\infty |V_{k'k_1k_2}|^2 f(k'k_1k_2) k'^2 k_1^2 k_2^2 \times \\ \times (n_{k_1} n_{k_2} + n_{k'} n_{k_2} - n_{k'} n_{k_1}) \delta_{\mathbf{k}2-\mathbf{k}_1-\mathbf{k}'} d\mathbf{k}_1 d\mathbf{k}_2. \end{aligned} \quad (13)$$

To obtain the Kolmogorov spectrum, one must solve the equation $P_k = P$. The solution of this equation has the form $\varepsilon_k = 4\pi \lambda c^{1/2} \rho_0^{1/2} P^{1/2} / k^{3/2}$ and coincides with formula (10).

To calculate the constant λ , let us use the approximate expression (7) for the kernel $V_{kk_1k_2}$.

After transformations we have

$$\frac{1}{\lambda^2} = \frac{9}{8} (1+q)^2 \int_0^\infty \frac{\ln(1+\xi)}{\xi^{5/2} (1+\xi)^{5/2}} [(1+\xi)^{9/2} - \xi^{9/2} - 1] d\xi,$$

which gives

$$\lambda \approx 0.2 / (1+q). \quad (14)$$

We note that the result $\varepsilon_k \sim k^{-3/2}$ (without calculation of coefficients) was obtained for weak turbulence in work ⁽¹⁾.

3. Let us now consider two-dimensional acoustic turbulence. All the considerations set out above concerning the character of the interaction of sound waves are also valid for the two-dimensional case. With a wave having wave vector $\mathbf{k}_0$, only waves whose wave vectors lie near k_0 within the angle $k_\perp^0/k_0$, where $k_\perp^0$ is determined by formula (5), will interact. However, the estimate for the time of nonlinear interaction will now be different. Obviously,

$$1/\tau \sim Va^* k_0 k_\perp^0 \sim (Va^*)^{3/2} k_0^3 \sim \varepsilon_k^{3/4}.$$

The formula for the energy spectrum in the two-dimensional case still has the form (9), but ρ_0 must be understood as the two-dimensional mass density. Using the result $1/\tau \sim \varepsilon^{3/4}$, we obtain

$$P \sim \varepsilon^{7/4}, \quad f(\xi) = \xi_0^{4/7}, \quad \varepsilon_k \approx \rho_0^{3/7} c^{2/7} P^{4/7} k^{-1/7}.$$

In the presence of dispersion one can also consider two-dimensional weak turbulence. However, it differs strongly from three-dimensional turbulence. Although the kinetic-

equation (11) and retains its form for two-dimensional turbulence, but when the δ -function of the wave numbers is averaged over angles there arises the factor $f_{kk_1k_2} = 2/\Delta(k, k_1, k_2)$, where $\Delta(k, k_1, k_2)$ is the area of the triangle formed by the vectors $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$. In the absence of dispersion, only waves directed along one straight line interact. For them $\Delta(k, k_1, k_2) = 0$, and the kernel of the kinetic equation becomes infinite. Therefore, in two-dimensional weak turbulence there is no limiting transition as $\varepsilon \rightarrow 0$.

Let us note that, for the applicability of the theory constructed above, the requirement of isotropy of the turbulence is not obligatory. It is sufficient that the distribution function $n(k)$ vary little over angles of the order of the characteristic interaction angle. This is all the more important because isotropization of the spectrum of acoustic turbulence must occur through processes of higher order in the nonlinearity than the establishment of a stationary state with respect to the modulus k , and may prove to be a very slow process.

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Institute of Nuclear Physics
Siberian Branch
Academy of Sciences of the USSR
Novosibirsk

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