

MIXED BOUNDARY-VALUE PROBLEMS OF GENERAL TYPE, NONSTATIONARY ON PART OF THE BOUNDARY

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Abstract

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MATHEMATICS

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MIXED BOUNDARY-VALUE PROBLEMS OF GENERAL TYPE, NONSTATIONARY ON PART OF THE BOUNDARY

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In this note we investigate certain problems for equations with partial derivatives which reduce to the Cauchy problem for an evolution equation of the type $du/dt = Au + F(t)$, where the operator A is generated by an elliptic eigenvalue problem with a spectral parameter in the boundary condition. In § 2 analogous problems are studied in which du/dt is replaced by other expressions, and in § 3 the case is considered of an operator A depending on t and $F = F(t, u)$.

1°. Let G be a bounded domain of n -dimensional space with smooth boundary $S = \Gamma_1 \cup \Gamma_2$. Consider the problem

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= -\mathcal{L}u(x, t) + f_0(x, t), \quad x \in G, \quad t > 0; \quad B'_j u(x, t) = 0, \quad x \in \Gamma_2; \\ \frac{\partial (C_j u)(x, t)}{\partial t} &= -B_j u(x, t) + f_j(x, t), \quad x \in \Gamma_1, \quad t > 0; \\ u(x, 0) &= \psi_0(x), \quad x \in G; \quad (C_j u)(x, 0) = \psi_j(x), \quad x \in \Gamma_1, \quad 1 \leq j \leq m. \end{aligned} \quad (1)$$

Here $\mathcal{L}$ is a properly elliptic differential expression of order $2m$; $\{B_j\}_{j=1}^m$ and $\{B'_j\}_{j=1}^m$ are normal systems of differential expressions of order m_j , covering $\mathcal{L}$ (B'_j may differ from B_j by coefficients and terms whose order is less than m_j); $\{C_j\}_{j=1}^m$ is a system of differential expressions of order $l_j = 2m - m_j - 1$, completing $\{B_j\}_{j=1}^m$ to a Dirichlet system of order $2m$ (see, for example, (1)). Problem (1) below is reduced to the Cauchy problem for an evolution equation.

We introduce the corresponding operator A . Let $C^{2m}(G \cup \Gamma_1)$ be the set of functions $2m$ times continuously differentiable in $G \cup \Gamma_1$; denote by L^2 the orthogonal sum of Hilbert spaces

$$\oplus \sum_{i=0}^m L_2(G^i);$$

x_0 ranges over $G^0 = G$ up to Γ_2 , and $x_1 \in G^k = \Gamma_1$, $1 \leq k \leq m$. The domain $D(A')$ of the operator A' consists of functions from $C^{2m}(G \cup \Gamma_1)$ such that $B'_j \varphi|_{\Gamma_2} = 0$, $1 \leq j \leq m$. The operator A' acts in the Hilbert space L^2 according to the rule

$$(\varphi(x_0), (C_1 \varphi)(x_1), \dots, (C_m \varphi)(x_1)) \rightarrow ((\mathcal{L} \varphi)(x_0), (B_1 \varphi)(x_1), \dots, (B_m \varphi)(x_1)). \quad (2)$$

The arguments of (2) allow one to prove the density of $D(A_1)$ in L^2 . If $\mathcal{L}$, $\{B_j\}_{j=1}^m$, and $\{B'_j\}_{j=1}^m$ are formally self-adjoint, and their coefficients and the boundary S are sufficiently smooth, then, according to Lemma 1 of (3), one can choose the system $\{C_j\}_{j=1}^m$ so that the operator A' will be symmetric. A' admits a closure in L^2 , which we denote by A . Let A^* be the operator in L^2 adjoint to A . Using the theorems on homeomorphisms (4), as in (2), one can prove that elements of $D(A)$ and $D(A^*)$ are regular and have the form

$$U(x) = (u(x_0), (C_1 u)(x_1), \dots, (C_m u)(x_1)).$$

The regularity properties

make it possible, by integration by parts, to obtain the equality $(A^* U, V)_{L^2} = (U, A^* V)_{L^2}$ ($U, V \in D(A^*)$), from which it follows that

Theorem 1. If $\mathcal{L}$, $\{B_j\}_{j=1}^m$ and $\{B'_j\}_{j=1}^m$ are formally self-adjoint, the surface S and the coefficients of the differential expressions are sufficiently smooth, and $\min_j m_j = m$, then the operator A is self-adjoint in L^2 .

This theorem generalizes Theorem 2 of (5).

Problem (1) in L^2 is equivalent to the Cauchy problem

$$\frac{dU(x, t)}{dt} = -AU(x, t) + F(x, t), \quad U(x, 0) = \Psi(x) \in L^2, \quad (3)$$

where $F(x, t) = (f_0(x_0, t), f_1(x_1, t), \dots, f_m(x_1, t))$, $\Psi(x) = (\psi_0(x_0), \psi_1(x_1), \dots, \psi_m(x_1))$, and $U(x, t) = (u(x_0, t), u_1(x_1, t), \dots, u_m(x_1, t))$ are regarded as functions of t with values in L^2 .

Theorem 2. Suppose the hypotheses of Theorem 1 are satisfied, and $\mathcal{L}$ and $\{B_j\}_{j=1}^m$ are such that A is nonnegative; $F(x, t)$ is strongly continuous for $t \in [0, \infty)$, and

$$\sup_{0 \leq t < \infty} \|F(x, t)\|_{L^2} < \infty.$$

Then, for $t > 0$, there exists a regular solution of problem (3) satisfying the initial condition $U(\cdot, t) \rightarrow \Psi(\cdot)$ as $t \rightarrow 0$ in the sense of strong convergence in L^2 . This solution is unique in L^2 , depends continuously (in the

sense of the L^2 norm) on $\Psi(\cdot)$, $F(\cdot, t)$, and is representable in the form $U(x) = (u(x_0, t), u_1(x_1, t), \dots, u_m(x_1, t))$, where

$$u_j(x_j, t) = \sum_{i=0}^m \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} e^{-\lambda t} \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)} d\sigma(\lambda) \right) \psi_i(y_i) dy_i + \\ + \sum_{i=0}^m \int_0^t \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} e^{-\lambda(t-\tau)} \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)} d\sigma(\lambda) \right) f_i(y_i, \tau) dy_i d\tau, \quad (4)$$

$$0 \leq j \leq m.$$

Here N_λ is the dimension of the eigensubspace corresponding to the spectral point λ ; $\varphi_\alpha(x, \lambda) = (\varphi_{\alpha 0}(x_0, \lambda), \varphi_{\alpha 1}(x_1, \lambda), \dots, \varphi_{\alpha m}(x_1, \lambda))$ is an eigenfunction of the operator A ; $\sigma(\lambda)$ is a positive measure.

We outline the **proof**. It is not hard to see that

$$U(\cdot, t) = e^{-At} \Psi(\cdot) + \int_0^t e^{-A(t-\tau)} F(\cdot, \tau) d\tau$$

is a solution of problem (3). The function

$$\gamma(A) = \int_0^\infty e^{-\lambda t} dE_\lambda$$

(E_λ is the resolution of the identity of the operator A) is bounded for $t \geq 0$ at large λ on the spectrum of the operator A ; therefore the method ⁽¹⁾, transferred to the orthogonal sum of Hilbert spaces in ⁽⁷⁾, gives representation (4). The operators $A^N e^{-At}$ ($N = 1, 2, \dots$) are bounded for $t > 0$; hence $U(x, t) \in D(A^N)$. The arguments of ⁽³⁾ make it possible to study the regularity properties of $D(A^N)$, from which the regularity of $U(x, t)$ follows.

Moreover, along these lines one can show that $\gamma(A)$ will be an integral operator with kernel $(K_{ij}(x_i, y_j, \lambda))_{i,j=0}^m$, and also study the singularities and regularity properties of the elements of the kernel and their connection with the eigenfunctions of the operator A .

We note that if the nonstationary boundary conditions are prescribed on the entire boundary of the domain, then the correctness of the problem considered follows from ⁽⁶⁾.

A theorem analogous to Theorem 2 can also be proved in the case when problem (1) is stationary inside the domain, i.e. the first equation in (1) is replaced by

$\mathcal{L}u(x, t) = 0$, while the remaining equations are retained. Similarly to §2 of (⁷), one can introduce an auxiliary self-adjoint operator in a Hilbert space

$$L^2(\Gamma_1) = \bigoplus_{i=1}^m \sum L_2(G^i)$$

(or its restrictions), an operator B closely connected with A . Then the problem under consideration in this Hilbert space (for definiteness, in $L^2(\Gamma_1)$) is equivalent to the Cauchy problem

$$\frac{d\hat{U}(x_1, t)}{dt} = -B\hat{U}(x_1, t) + F_1(x_1, t), \quad \hat{U}(x_1, 0) = \Psi(x_1) \in L^2(\Gamma_1);$$

$$\hat{U}(x_1, t), F_1(x_1, t) \in L^2(\Gamma_1).$$

There holds

Theorem 3. *If the conditions of Theorem 1 are satisfied, the nonnegative operator A has a bounded inverse, $F_1(x_1, t)$ is strongly continuous for $t \in [0, \infty)$, and*

$$\sup_{0 \leq t < \infty} \|F_1(x_1, t)\|_{L^2(\Gamma_1)} < \infty,$$

then problem (1), stationary in the interior of the domain, is well posed, its solution is regular and is representable analogously to (4).

2°. Problem (1) for the Schrödinger equation has, for example, the form

$$\frac{1}{i} \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) - c(x)u(x, t), \quad x \in G, \quad t > 0; \quad \frac{\partial u}{\partial n} + \sigma(x)u|_{\Gamma_2} = 0;$$

$$\frac{1}{i} \frac{\partial u(x, t)}{\partial t} = -\frac{\partial u(x, t)}{\partial n}, \quad x \in \Gamma_1, \quad t > 0; \quad u(x, 0) = \psi_0(x), \quad x \in G;$$

$$u(x_1, 0) = \psi_1(x_1), \quad x_1 \in \Gamma_1$$

($\sigma(x)$ and $c(x)$ are nonnegative real functions; Δ is the Laplace expression; n is the exterior normal). In this case the operator A is not semibounded; however, Theorems 2 and 3 are valid with $e^{-\lambda t}$ replaced by $e^{-i\lambda t}$ and $F(x, t) \equiv 0$.

Under the assumptions of the preceding item, the hyperbolic problem

$$\frac{\partial^2 u(x, t)}{\partial t^2} = -\mathcal{L}u(x, t) + f_0(x, t), \quad x \in G, \quad t > 0; \quad B'_i u(x, t) = 0, \quad x \in \Gamma_2;$$

$$\frac{\partial^2 (C_j u)(x, t)}{\partial t^2} = -B_j u(x, t) + f_j(x, t), \quad x \in \Gamma_1, \quad t > 0; \quad (5)$$

$$u(x, 0) = \psi_0(x), \quad x \in G; \quad (C_j u)(x, 0) = \psi_j(x), \quad x \in \Gamma_1;$$

$$\frac{\partial u(x, 0)}{\partial t} = \varphi_0(x), \quad x \in G; \quad \frac{\partial (C_j u)(x_1, 0)}{\partial t} = \varphi_j(x_1), \quad x_1 \in \Gamma_1; \quad 1 \leq j \leq m$$

is equivalent to the Cauchy problem

$$\frac{d^2 U(x, t)}{dt^2} = -AU(x, t) + F(x, t), \quad t > 0;$$

$$U(x, 0) = \Psi(x), \quad \frac{\partial U(x, 0)}{\partial t} = \Phi(x).$$

Theorem 4. Let $\Psi(x) \in D(A)$, $\Phi(x) \in D(A^{1/2})$, and let $F(x, t)$ be continuously differentiable with respect to t , and for every $t \in [0, \infty)$ let $F(x, t) \in D(A^{1/2})$. Then (5) has a generalized solution (from $\widetilde{W}_{x,t,2}^{m,2}$, defined in (4)), which is unique in L^2 , depends continuously (in the sense of the L^2 -norm) on the initial data, and is representable in the form $U(x, t) = (u(x_0, t), u_1(x_1, t), \dots, u_m(x_1, t))$, where for each $0 \leq j \leq m$

$$\begin{aligned} u_j(x_j, t) = & \sum_{i=0}^m \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} \cos \lambda^{1/2} t \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)} d\sigma(\lambda) \right) \psi_i(y_i) dy_i + \\ & + \sum_{i=0}^m \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} \lambda^{-1/2} \sin \lambda^{1/2} t \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)} d\sigma(\lambda) \right) \varphi_i(y_i) dy_i + \\ & + \sum_{i=0}^m \int_0^t \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} \lambda^{-1/2} \sin \lambda^{1/2} (t - \tau) \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)} d\sigma(\lambda) \right) f_i(y_i, \tau) dy_i d\tau. \end{aligned}$$

If A satisfies the conditions of Theorem 3, then the assertions of this theorem are also valid in the case when the first equation of problem (5) is

$$\mathcal{L}u(x, t) = 0.$$

In this way one can also study more general problems. For example, problems that are equivalent to the Cauchy problem

$$L(t)U(x, t) = -AU(x, t) + F(x, t), \quad U^{(i)}(x, 0) = \Psi_i(x), \quad x \in G \cup \Gamma_1, \\ i = 0, 1, 2, \dots, k-1,$$

where $L(t)$ is a differential operator in t of order k .

3°. When the coefficients of $\mathcal{L}$ and $\{B_j\}_{j=1}^m$ depend on t , then $A = A(t)$ and $B = B(t)$. Usually, if the coefficients of the boundary expressions depend on t , then the domain of definition of the operator also depends on t , which significantly complicates the study of the Cauchy problem. In our case, the construction of $A(t)$ and $B(t)$ implies independence of t for their domains of definition.

Further, one may consider $\mathcal{L}$ and B_j containing nonlinear terms $f_i(t, u_i(x_i))$ ($0 \leq i \leq m$); then the operator equation will also contain the nonlinear term

$$F(t, U) = (f_0(t, u(x_0, t)), f_1(t, u_1(x_1, t)), \dots, f_m(t, u_m(x_m, t))).$$

Therefore, in these cases as well one can obtain results analogous to (6, 8–11) for boundary-value problems of the type indicated above.

4°. Generalizing (5), in a number of cases one can prove the self-adjointness of the operators A and B also for unbounded domains. This makes it possible to consider problems analogous to those given above also in unbounded G . The theory of semigroups also permits the study of a broader class of problems, to which non-self-adjoint but maximally dissipative (see (9)) operators correspond.

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Note: Figure translations are in progress. See original paper for figures.

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