

$\backslash(P\backslash)$ -SETS IN TOPOLOGICAL SPACES

A. I. Veksler

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.87152>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.88:513.83+519.50+519.54

MATHEMATICS

A. I. VEKSLER

P-SETS IN TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov on 16 I 1970)

1. We shall, as far as possible, adhere to the terminology of the monographs^(3,4) and the survey article⁽¹⁾.

The concept of a *P*-point belongs to L. Gillman and M. Henriksen⁽⁵⁾. A *P*-point is a point of a completely regular space T such that, in the intersection of any countable number of (open) neighborhoods of it, there is still one neighborhood.

It is well known, for example, that an arbitrary $t \in T$ is a *P*-point if and only if every $x \in C(T)$ is constant in some neighborhood of t . If t is a *P*-point in T and $t \in A \subset T$, then t is a *P*-point also in A . A point $t \in T$ is a *P*-point if and only if there exists only one prime ideal in the ring $C(T)$ attached to t , namely the ideal M_t of all functions vanishing at t .

A natural generalization of the concept of a *P*-point is the concept of a *P*-set. A closed set $F \subset T$ is called a *P*-set if, in the intersection of any countable number of open sets containing F , there is still one such set.

For simplicity we shall consider *P*-sets only in bicompacts and assume that all spaces considered below (but not their subspaces) are bicompacts.

For an arbitrary bicompact K , its *P*-sets can be characterized as follows: $F \subset K$ is a *P*-set if and only if it is closed and every $x \in C(K)$ that is constant on F is constant also on some neighborhood of F . A closed Φ is a *P*-set if and only if

$$I_\Phi = \{x \in C(K) : x(\Phi) \equiv 0\}$$

is the unique ideal of $C(K)$ for which $\{M_k : k \in \Phi\}$ is the collection of all maximal ideals containing this ideal.

Although *P*-sets had appeared episodically earlier in some works, the value of the very concept of a *P*-set was first, in our opinion, shown in the work of Z. T. Dikanova⁽⁶⁾. Z. T. Dikanova considered *P*-sets in extremally disconnected bicompacts* and showed that *P*-sets arise naturally in the theory of semiordered spaces when solving certain problems concerning extended K -spaces. Recall that extended K -spaces are precisely those K -spaces that are isomorphic to spaces $C_\infty(Q)$ of all continuous real functions on an extremally disconnected bicompact, which may take the values $\pm\infty$ on nowhere dense subsets of Q .

Apparently, it may now be regarded as a timely problem to study P -sets independently. Here, however, this problem is not considered. In this note, first, it is shown that the consideration of P -sets in quasiextremal bicomacts is natural in solving certain other problems of the theory of semiordered spaces, essentially different from

* Z. T. Dikanova additionally required that a P -set be nowhere dense. Under such a definition, however, an isolated point is not a P -set, although it is a P -point.

problems solved in (6) and relating to extended K_σ -spaces. Recall that a bicompactum S is called **quasiextremal** (or basically disconnected) if in it the closure of every open set of type F_σ is open; of course, an extremally disconnected bicompactum is quasiextremal. Every **extended K_σ -space** is isomorphic to the space $C_\infty(S)$, and conversely, every function space of this type is an extended K_σ -space. Secondly, with the aid of certain results relating to extended K_σ -spaces and obtained earlier by the author, some properties of P -sets of an arbitrary quasiextremal bicompactum will be derived very simply.

2. A positive linear mapping φ of one K -linear (vector-lattice) structure X onto another Y is called a **structural homomorphism** if

$$\varphi(x_1 \vee x_2) = \varphi(x_1) \vee \varphi(x_2) \quad \text{and} \quad \varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2).$$

If I is an l -ideal which is the kernel of the homomorphism φ , then the K -lineals Y and X/I are naturally isomorphic. A K -lineal Z is called **Archimedean** if from the inequalities $0 \leq nz \leq z'$ ($n \in N$) it follows that $z = 0$; of course, a K_σ -space is Archimedean. In (2) the following was proved.

Theorem 0. Let X be an extended K_σ -space, and I its l -ideal. In order that the K -lineal X/I be Archimedean, it is necessary and sufficient that I be σ -closed, i.e., from $\{x_n\} \subset I$ ($n \in N$) and the existence of

$$x = \sup x\{x_n\}$$

it follows that $x \in I$. When this condition is fulfilled, X/I is an extended K_σ -space.

Let us now translate the first part of this theorem into topological language. In what follows, by the **support** of $x \in C_\infty(S)$ on a quasiextremal S we shall mean the open-closed set

$$S_x = \{s \in S : x(s) \neq 0\}.$$

Theorem 1. Let S be a quasiextremal bicompactum, $X = C_\infty(S)$, and let I be an l -ideal in X . In order that the K -lineal X/I be Archimedean, it is necessary and sufficient that

$$I = I_F = \{x \in X : x(F) \equiv 0\}$$

for some P -set F .

Proof. It is necessary to verify that the condition of the theorem is equivalent to the σ -closedness of the l -ideal I .

Let I be a σ -closed l -ideal. As the required F take the intersection of the zero-sets of all functions in I . Let $e \in X$ be the unit (i.e. the function taking only the values 0 and 1) and let $e(F) \equiv 0$. We show that $e \in I$. By assumption, for any $s \in S_e \subset S \setminus F$ there exists $x_s \in I$ such that $x_s(s) \neq 0$. Since I is an l -ideal, without loss of generality we may suppose that $x_s > 0$ and $x_s(s) > 1$. From the cover of the closed set S_e by the open sets

$$S_s = \{s' \in S : x_s(s') > 1\}$$

choose a finite subcover $S_{s_1}, \dots, S_{s_n}$. Obviously,

$$0 \leq e \leq \sup\{x_{s_i}\} \in I,$$

whence $e \in I$.

Let us check that $I = I_F$. It is enough to make sure that if $x \in X^+$ and $z(F) \equiv 0$, then $z \in I$. It is known that any $z \in X^+$ can be represented as the supremum of a sequence of positive linear combinations z_n of unit elements from X . But, obviously, in the present case the supports of these unit elements do not intersect F , i.e. these unit elements (and then also z_n) by what has already been proved belong to I . And then

$$z = \sup\{z_n\} \in I$$

by virtue of the σ -closedness of I . Let us verify that F is a P -set.

Let $\{G_n\}$ be a sequence of open neighborhoods of the set F . We check that there exists an open neighborhood contained in each G_n . By virtue of the complete disconnectedness of S , it suffices to restrict ourselves to the case of open-closed G_n . Let e_n be the characteristic function of the set $S \setminus G_n$. By what has already been proved, $e_n \in I$. Let further $e = \sup\{e_n\}$ and

$$G = S \setminus S_e.$$

Then G is open-closed,

$$G = \text{Int}\left(\bigcap G_n\right),$$

and $e_n \in I$ by virtue of the σ -closedness of I . Hence $F \subset G$. Thus F is a P -set.

We prove also conversely that every l -ideal $I = I_F$, where F is a P -set, is σ -closed. If $z \in I^+$, then

$$S_z = \bigcup \{S_m : m \in N\},$$

where

$$S_m = \{s \in S : z(s) > m^{-1}\}.$$

But since the S_m are closed and $S_m \cap F = \Lambda$, then also

$S_z \cap F = \Lambda$ (in view of the fact that F is a P -set). Now let $\{z_n\} \subset I^+$ and $x = \sup\{z_n\}$. Then $S_{z_n} \cap F = \Lambda$ and $S_z = \bigcup S_{z_n}$. Again, since F is a P -set, from the last two equalities it follows that $S_z \cap F = \Lambda$, i.e. $z(F) = 0$, $z \in I$. The theorem is proved.

We see that the problem of finding structurally homomorphic Archimedean images of the extended K_σ -space X reduces to the problem of finding all σ -closed l -ideals in X ; the second problem turns out to be equivalent to the problem of finding all σ -closed l -ideals in $C_\infty(S)$, and the latter problem turns out to be equivalent to the problem of finding all P -sets in the quasiextremal bicom pactum S . Thus a problem from the theory of partially ordered spaces turns out to be equivalent to the topological problem of finding all P -sets of a quasiextremal bicom pactum.

3. With the aid of Theorems 0 and 1 we shall establish a number of properties of P -sets in a quasiextremal bicom pactum S . Some of them were proved by Z. T. Dikanova by a direct method for the case of an extremally disconnected S in ⁽⁶⁾, or in her report at the seminar at Leningrad University. We shall call open-and-closed sets **trivial** P -sets, and the remaining P -sets **nontrivial**.

Theorem 2. *For a quasiextremal bicom pactum the following assertions hold:*

- a) *Nontrivial P -sets in S exist if and only if S is of uncountable type (in other terminology, when its cellularity number is greater than $\aleph_0$).*
- b) *If $\{E_\xi\}$ ($\xi \in \Xi$) is a system of P -sets from S , then also $F = \bigcup \overline{F_\xi}$ is a P -set.*
- c) *Every closed Φ contains the greatest P -set from S .*
- d) *The family of all P -sets from S , partially ordered by inclusion, is a complete lattice with least element Λ and greatest element S .*
- e) *A closed Φ is a P -set if and only if the trace of every closed nowhere dense (c.n.d.) set of type G_δ from S is n.n.d. also in Φ (and, obviously, is a closed set of type G_δ in Φ).*
- f) *A closed Φ is a P -set if and only if, for no open-and-closed $S_0 \subset S$ for which $S_0 \cap \Phi \neq \Lambda$, does this intersection immerse itself in a c.n.d. set from S of type G_δ .*

Proof. a) In Theorem 1, by means of the equality $I = I_F$, a one-to-one correspondence was established between σ -closed l -ideals in the K_σ -space $X = C_\infty(S)$ and P -sets in S . If F is open-and-closed, then this precisely means that I_F is a component in X , i.e. a closed l -ideal. Thus the existence of nontrivial P -sets in S is equivalent to the existence of σ -closed l -ideals in X that are not components. But if S is of countable type, then X is also of countable type, and then every σ -closed l -ideal is a component in X . Hence in this case there are no nontrivial

P -sets in S . Conversely, if S has uncountable type, then let $\{S_\xi\}$ ($\xi \in \Xi$) be an uncountable system of nonempty pairwise disjoint open-and-closed sets from S . Put $I_\xi = I_{S_\xi}$, and let I be the l -ideal whose elements are all suprema in X of finite or countable sets from I_ξ ($\xi \in \Xi$). From the σ -closedness of all I_ξ follows the σ -closedness of I . But, obviously, I is not a component. Therefore, if $I = I_F$, then F will be a nontrivial P -set in S .

- b) We have $I_F = \bigcap I_{F_\xi}$. But the intersection of any set of σ -closed l -ideals is itself a σ -closed l -ideal. Hence F is a P -set.
- c) Follows from b).
- d) Follows from b) and c).
- e) Let $I = I_\Phi$. Then X/I is isomorphic to the K -linear $C_\infty^\Phi(S)$ of functions that are traces on Φ of functions from $C_\infty(S)$. It is clear that $C_\infty(\Phi) \subset C_\infty^\Phi(S)$.

A K -linear $C_\infty^\Phi(S)$ may fail to be Archimedean if (and only if) the set of points at infinity of some $y \in C_\infty^\Phi(S)$ (it is always G_δ in Φ) is not an n.n.p. in Φ . But the collection of all sets of points at infinity of functions from $C_\infty^\Phi(S)$ is precisely the collection of traces on Φ of all z.n.n.p. sets in S of type G_δ (each such set in S is the set of points at infinity of a function from $C_\infty(S)$). This implies d).

Let us note that from the proof it is easy to derive that Φ is a P -set if and only if $C_\infty(\Phi) = C_\infty^\Phi(S)$.

We omit the straightforward proof of e), observing only that d) and e) are equivalent for any completely disconnected bicom pactum.

- 4. We now translate the second part of Theorem 0 into topological language and obtain from it some corollaries.

Theorem 3. *A P -set F of a quasi-extremal bicom pactum is itself quasi-extremal.*

Proof. Let $I = I_F$. Then $X/I = C_\infty^F(S)$, and, by the remark made in the proof of Theorem 2d), $X/I = C_\infty(F)$. Since F is a P -set, X/I is a K_σ -space. Hence F is quasi-extremal.

Theorem 4. *If S is a quasi-extremal bicom pactum, F is a P -set in S , and Φ is a P -set in F , then Φ is a P -set also in S .*

Proof. Obviously,

$$X/I = C_\infty(S)/I_\Phi = C_\infty(\Phi)/I_\Phi^F = C_\infty(\Phi).$$

Here I_Φ^F is the l -ideal of the functions I_Φ , but considered in $C_\infty(F)$, and the last equality follows from the fact that Φ is a P -set in F . By Theorem 3, Φ is quasi-extremal; then $C_\infty(\Phi)$ is a K_σ -space. By Theorem 1, Φ is a P -set in S .

Let us consider one application of the last result to extremally disconnected bicom pacta. It is known that in an extremally disconnected bicom pactum Q there

are nonisolated P -points if and only if the cardinality of its type is measurable. An obvious consequence of Theorem 3 is the following statement.

Corollary 1. *If Q is an extremally disconnected bicompactum whose type cardinality (number of cellularity) is nonmeasurable and in which there are no isolated points, and F is a P -set in Q , then the bicompactum F has no P -points (and, in particular, no isolated points).*

If the question of the existence of nontrivial quasi-extremal P -sets in a quasi-extremal bicompactum is completely solved by Theorems 2a) and 3, then the question of the existence of nontrivial extremally disconnected P -sets in an extremally disconnected bicompactum Q is considerably more difficult. This question was formulated by R. Sikorski ⁽⁷⁾. It is equivalent to an old question in the theory of partially ordered spaces. Namely, can one find, in an extended K -space, an l -ideal I which is not a component, for which X/I is an (extended) K -space? Of course, if the type cardinality of Q is measurable, then Q even has nontrivial P -points. However, it is not known whether there exists a Q of nonmeasurable type cardinality having a nontrivial extremally disconnected P -set.

Leningrad Institute
of Textile and Light Industry
named after S. M. Kirov

Received
16 I 1970

CITED LITERATURE

1. P. S. Aleksandrov, UMN, 19, No. 6, 3 (1964).
2. A. I. Veksler, Scientific Notes of the Leningrad State Pedagogical Institute named after A. I. Herzen, 183, 107 (1958).
3. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, 1961.
4. L. Gillman, M. Jerison, *Rings of Continuous Functions*, 1960.
5. L. Gillman, M. Henriksen, Trans. Am. Math. Soc., 77, No. 2, 340 (1954).
6. Z. T. Dikanova, Siberian Mathematical Journal, 9, No. 4, 804 (1968).
7. R. I. Sikorski, Proceedings of the IV All-Union Topological Conference, 1967, p. 170.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.