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Abstract

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MATHEMATICS

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ON THE CONFORMAL EQUIVALENCE OF THE CARLEMAN BOUNDARY-VALUE PROBLEM TO THE RIEMANN BOUNDARY- VALUE PROBLEM ON AN OPEN CONTOUR

(Presented by Academician N. I. Muskhelishvili on 26 V 1969)

1. Let D be a finite $(m+1)$ -connected domain bounded by a Lyapunov contour L , consisting of a curve L_0 enclosing the curves $L_1, \dots, L_m$. We assume that the origin belongs to the domain D .

By $\omega(z, L_k)$, $k = 0, \dots, m$, we denote the harmonic measures of the boundary curves L_k with respect to the domain D (¹), and by $D_0^-, \dots, D_m^-$ the complement of the domain $D + L$ to the complete plane.

Let

$$\alpha(t) = \sum_{k=0}^m \omega(t, L_k) \alpha_k(t),$$

where $\alpha_k(t)$ is an orientation-changing homeomorphism of the contour L_k onto itself, satisfying the following two conditions: 1) $\alpha_k[\alpha_k(t)] \equiv t$ (Carleman condition); 2) the function $\alpha'_k(t) \in H(L)^*$, $\alpha'_k(t) \neq 0$, $t \in L$. Let $G(t) \in H(L)$, $g(t) \in H(L)$, $G(t) \neq 0$, $t \in L$.

Consider the following problem.

Find a function $\Phi(z)$, single-valued and analytic in D , H -continuous in $\bar{D}$, satisfying one of the following conditions on L :

$$\Phi^+[\alpha(t)] = G(t)\Phi^+(t), \quad (1)$$

$$\Phi^+[\alpha(t)] = G(t)\Phi^+(t) + g(t). \quad (2)$$

Only the case in which the conditions $(2,3)$ are fulfilled is of interest:

$$G[\alpha(t)]G(t) = 1, \quad G[\alpha(t)]g(t) + g[\alpha(t)] = 0. \quad (3)$$

The boundary-value problem (1) was first posed by T. Carleman ⁽⁴⁾. A complete solution of problems (1), (2) for a bounded simply connected domain was given by D. A. Kveselava ⁽²⁾. The case of a simply connected unbounded exterior domain was investigated by T. S. Litvinchuk ⁽³⁾. Problem (1) for a multiply connected domain was first considered by E. I. Zverovich ^(5,6), who computed the number of solutions of problem (1) for the cases $\chi > 0$ and $\chi < -2(m-1)$, where $\chi = \text{Ind } G(t)|_L$. However, the method applied in ⁽⁵⁾ did not provide an algorithm for finding the solutions of problem (1) and proved insufficient for finding the number of solutions of the problem in the case $-2(m-1) \leq \chi \leq 0$.

In the present note we establish the fact of the conformal equivalence of the Carleman problem to the Riemann problem on a contour consisting of $m+1$ open arcs. This is achieved by constructing a special gluing function. Up to now, what was known was the conformal equivalence, established by G. F. Mandzhavidze and B. V. Khvedelidze ⁽⁷⁾ and by I. B. Simonenko ⁽⁸⁾, of the Haseman problem to the Riemann problem (on a closed contour); a problem of the type of the Haseman problem was included by E. I. Zverovich ⁽⁵⁾ in the general

* That is, satisfying the Hölder condition (H -continuous) on L .

theory of the Riemann problem on abstract Riemann surfaces; the equivalence of a problem of Carleman type to the Hilbert problem was proved by G. S. Litvinchuk and E. G. Khasabov [9]. The Carleman problem (2) remained the only one of the principal boundary-value problems with shift*, for which such a fact had not hitherto been known.

The connection established between the Carleman problem and the Riemann problem on an open contour makes it possible to give a complete solution of the Carleman problem for a multiply connected domain.

2. Lemma 1. A function $\Phi(z)$, single-valued and analytic in D , H -continuous in $\overline{D}$, can be represented in the form:

$$\Phi(z) = \frac{1}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} d\tau + \int_L \omega(\tau, L_m) \varphi(\tau) [1 + |\alpha'(\tau)|] d\sigma, \quad (4)$$

where the density $\varphi(t)$ satisfies the condition

$$\varphi(t) + \varphi[\alpha(t)] = \sum_{k=1}^m c_k \omega(t, L_k); \quad (5)$$

here c_k , $k = 1, \dots, m-1$, are arbitrary constants, while c_m is determined by $\Phi(z)$ uniquely; moreover the density $\varphi(t)$ is determined up to a term of the form $\sum_{k=1}^{m-1} \mu_k \omega(t, L_k)$, where μ_k are arbitrary constants.

Lemma 2. The general solution of the problem

$$\Phi^+[\alpha(t)] = \Phi^+(t) \quad (6)$$

is an arbitrary constant.

Using representation (4), we reduce problem (6) to the integral equation

$$\varphi(t) + \frac{1}{2\pi i} \int_L \left[\frac{1}{\tau - t} - \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)} \right] \varphi(\tau) d\tau = 0. \quad (7)$$

It is easy to show that every solution of equation (7) satisfies condition (5); the functions $\omega(t, L_1), \dots, \omega(t, L_m)$ are eigenfunctions of equation (7), and to the functions $\omega(t, L_k)$, $k = 1, \dots, m-1$, there corresponds the trivial solution of problem (7), while the function $\omega(t, L_m)$ corresponds to a complex constant.

Consider the equation

$$\psi(t) - \frac{1}{2\pi i} \int_L \left[\frac{1}{\tau - t} - \frac{\alpha'(t)}{\alpha(\tau) - \alpha(t)} \right] \psi(\tau) d\tau = 0, \quad (8)$$

conjugate to equation (7). For a fundamental system of solutions $\psi_1(t), \dots, \psi_m(t)$ of equation (8), the conditions

$$\psi_k(t) + \psi_k[\alpha(t)]\alpha'(t) = 0, \quad k = 1, \dots, m. \quad (9)$$

are fulfilled. Hence follows the unconditional solvability of the problem**

$$\Phi^+[\alpha(t)] = \Phi^+(t) + g(t) \quad (g(t) + g[\alpha(t)] = 0). \quad (10)$$

3. The following theorem on conformal welding is based on the last assertion, under the assumption that $g(t) = 1/t - 1/\alpha(t)$.

Theorem 1. In the domain D there exists a univalent function $\omega^+(z)$, analytic in D , except for the point $z = 0$, where $\omega^+(z)$ has a simple

* Here and above we use the terminology of the survey article [6].

** It is easy to see that the solvability condition for problem (10) $\int g(t)\psi_k(t) dt = 0$, $k = 1, \dots, m$, is satisfied

strip satisfying on L the gluing condition

$$\omega^+[\alpha(t)] = \omega^+(t). \quad (11)$$

Moreover, the gluing line Γ consists of $m + 1$ simple open Lyapunov contours $\Gamma_0, \dots, \Gamma_m$, given by the equation

$$w = \omega^+(t), \quad t \in L. \quad (12)$$

The function $\omega^+(z)$ has the form $\omega^+(z) = 1/z + \Phi^+(z)$. It is not difficult to show that $\omega^+(z)$ is univalent.

4. Let us introduce a new unknown function

$$\Psi(\omega) = \Phi[z(\omega)], \quad (13)$$

where $z(\omega)$ is the function inverse to $\omega^+(z)$. Then problem (2) reduces to two Riemann problems on the open contour Γ , consisting of $m + 1$ arcs. Both problems, by virtue of conditions (3), turn out to be identical. Thus, the Riemann problem corresponding to the Carleman problem has the form

$$\Psi^+(w) = G[z^-(w)]\Psi^-(w) + g[z^-(w)], \quad w \in \Gamma. \quad (14)$$

The index of problem (14) is represented by formula (10)

$$\nu = -(\chi + m_-)/2, \quad (15)$$

where m_- is the number of fixed points of the shift $\alpha(t)$ at which $G(t) = -1$, $\chi = \text{Ind } G(t)|_L$.

Relying on the theory of the Riemann problem on an open contour ⁽¹⁰⁾, we obtain the result for the Carleman problem in the case of an $(m + 1)$ -connected domain.

Theorem 2. The number of solutions of the homogeneous Carleman problem (1) is $l = 1 - (\chi + m_-)/2$ for $\chi \leq -m_-$, where $\chi = \text{Ind } G(t)|_L$, and m_- is the number of fixed points of the shift $\alpha(t)$ at which $G(t) = -1$.

For $\chi > -m_-$, the homogeneous problem (1) has no nontrivial solutions.

For $\chi > -m_-$, the nonhomogeneous problem (2) is solvable and has a unique solution if $p = (\chi + m_-)/2 - 1$ conditions are fulfilled.

In the case $m = 0$, the known theorem of D. A. Kveselava is obtained ^(2,6).

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