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Abstract

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HYDROMECHANICS

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THE PROBLEM OF GENERATION OF A MAGNETIC FIELD IN THE PRESENCE OF ACOUSTIC TURBULENCE

(Presented by Academician R. Z. Sagdeev on 3 III 1970)

One of the basic problems of magnetohydrodynamic turbulence consists in clarifying the question of whether a turbulent medium is stable with respect to weak perturbations of the magnetic field, i.e., whether weak magnetic fields will grow with time. The main difficulty is the absence of a small parameter in ordinary hydrodynamic turbulence ⁽¹⁾. Such a small parameter can be found in acoustic turbulence, namely $\tau v/\lambda$, where τ is the correlation time at one point, λ is the characteristic wavelength, and v is the characteristic velocity.

In problems of generation of the magnetic field $\mathbf{H}$, the velocity field $\mathbf{v}$ is assumed to be prescribed, while $\mathbf{H}$ satisfies the equation

$$\partial \mathbf{H} / \partial t = \text{rot}[\mathbf{v}, \mathbf{H}] + v_m \Delta \mathbf{H}, \quad (1)$$

where v_m is the magnetic viscosity.

I. We shall assume the velocity field to be stationary and distributed homogeneously and isotropically in space. It is convenient to pass to $k - \omega$ -space

$$\mathbf{v} = \int \frac{\mathbf{k}}{k} \varphi(\mathbf{k}, \omega) e^{i[(\mathbf{k}\mathbf{r}) - \omega t]} d\mathbf{k} d\omega.$$

Next, averaging over the ensemble, we obtain

$$\langle u_i(\mathbf{k}, \omega) u_j^*(\mathbf{k}', \omega') \rangle = \frac{k_i k_j}{k^2} I(k, \omega) \delta(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega'); \quad (2)$$

$$\langle v_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t + s) \rangle = \int \frac{k_i k_j}{k^2} f(k, s) e^{i(\mathbf{k}\mathbf{r})} d\mathbf{k};$$

$$f(k, s) = \int I(k, \omega) e^{-i\omega s} d\omega. \quad (3)$$

Let $E(k)$ be the power spectrum of the acoustic oscillations; then $E(k) = 4\pi k^2 f(k, 0)$. The correlation time may be introduced in the following way:

$$\tau = \int dk \int_0^\infty ds f(k, s) / \int dk f(k, 0) = \pi \int dk I(k, 0) / \int dk f(k, 0). \quad (4)$$

We now determine $I(k, 0)$. For this purpose we represent the density ρ and the velocity as follows: $\rho = \rho_0 + \rho_1 + \rho_2$; $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ ($\rho_0 \gg \rho_1 \gg \rho_2$, $\mathbf{v}_1 \gg \mathbf{v}_2$). The Fourier images in k - ω -space of the quantities $\rho_1, \rho_2, \mathbf{v}_1, \mathbf{v}_2$ will be denoted by $\rho_1(\mathbf{k}, \omega), \rho_2(\mathbf{k}, \omega), \frac{\mathbf{k}}{k}\varphi_1(\mathbf{k}, \omega), \frac{\mathbf{k}}{k}\varphi_2(\mathbf{k}, \omega)$. Linearization of the equations of gas dynamics gives, as is known, a solution in the form of acoustic waves, with

$$\rho_1(\mathbf{k}, \omega) = \frac{1}{c} \rho_0 \varphi_1(\mathbf{k}, \omega);$$

$$\langle \varphi_1(\mathbf{k}, \omega) \varphi_1^*(\mathbf{k}', \omega') \rangle = \delta(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega') f(k, 0) \frac{1}{2} [\delta(\omega - ck) + \delta(\omega + ck)] \quad (5)$$

(c is the speed of sound). To obtain $I(k, 0)$ we write the equation for the correction of second approximation (it is sufficient to use only the equation—continuity equation):

$$\rho_0 k \varphi_2(\mathbf{k}, \omega) = \omega \rho_2(\mathbf{k}, \omega) - \int \rho_1(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) \frac{(\mathbf{k} \mathbf{k}_1)}{k_1} \varphi_1(\mathbf{k}_1, \omega_1) d\mathbf{k}_1 d\omega_1. \quad (6)$$

Multiplying (6) by $\varphi_2^*(\mathbf{k}', \omega')$, using (5), and averaging, the expression on the right-hand side of (6), when multiplied by the complex conjugate, then gives a fourth-order moment. Here one may use the random-phase approximation and integrate over $d\mathbf{k}'$ and $d\omega'$; then we obtain ($\mathbf{p} = \mathbf{k} - \mathbf{q}$):

$$\begin{aligned} I(\mathbf{k}, 0) &= \frac{1}{2c^2 k^2} \int \left[\frac{(\mathbf{k} \mathbf{q})^2}{q^2} + \frac{(\mathbf{k} \mathbf{q})(\mathbf{k} \mathbf{p})}{qp} \right] f(p, 0) f(q, 0) \delta(cp - cq) d\mathbf{q} = \\ &= \frac{\pi k}{2c^3} \int_{k/2}^\infty f^2(q, 0) dq; \\ \pi &= \frac{\pi^2}{16c^3} \int k^2 f^2\left(\frac{k}{2}, 0\right) dk / \int f(k, 0) dk. \end{aligned} \quad (7)$$

In order of magnitude $\tau \approx (v^3/c^3)\lambda/v$, so that $\tau \ll \lambda/v$.

II. To derive the equation for the spectral function of the magnetic field $B(k, t)$, we shall use the Fourier representation (1) in k -space

$$\mathbf{H}(\mathbf{k}, t) = \mathbf{H}(\mathbf{k}, 0)e^{-k^2\nu_m t} + i \int_0^t dt_1 e^{-k^2\nu_m(t-t_1)} \int d\mathbf{q} \left[\mathbf{k} \left[\frac{\mathbf{p}}{p} \psi(\mathbf{p}, t_1) \mathbf{H}(\mathbf{q}, t_1) \right] \right]. \quad (8)$$

$\mathbf{H}(\mathbf{k}, t)$ and $\psi(\mathbf{k}, t)\mathbf{k}/k$ are the Fourier transforms of $\mathbf{H}(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$.

We use a perturbation-theory expansion in the velocity:

$$\mathbf{H}(\mathbf{k}, t) = \sum_{n=0}^{\infty} \mathbf{H}^{(n)}; \quad \mathbf{H}^{(0)} = \mathbf{H}(\mathbf{k}, 0)e^{-\nu_m k^2 t},$$

$$\mathbf{H}^{(n)} = i \int_0^t dt_1 e^{-\nu_m k^2(t-t_1)} \int d\mathbf{q} \left[\mathbf{k} \left[\frac{\mathbf{p}}{p} \psi(\mathbf{p}, t_1) \mathbf{H}^{(n-1)}(\mathbf{q}, t_1) \right] \right]. \quad (9)$$

We restrict ourselves to terms of second order in ψ (which is justified when $\tau \ll \lambda/v$). Multiplying (9) by $H_j^*(\mathbf{k}', t)$, we shall assume: 1) $\mathbf{H}(\mathbf{k}, 0)$ is statistically independent of $\psi(\mathbf{k}, t)$; 2) ν_m is small, so that in (9) one may put $\exp(-\nu_m k^2 t) = 1 - \nu_m k^2 t$; 3) $\langle H_i(\mathbf{k}, 0) H_j^*(\mathbf{k}', 0) \rangle = B(k, 0) \delta(\mathbf{k} - \mathbf{k}') \times (\delta_{ij} - k_i k_j / k^2)$.

Averaging the resulting expression and integrating over $d\mathbf{k}'$, we obtain

$$B(k, t) = B(k, 0) - 2B(k, 0) \left[\int_0^t ds (t-s) \int f(q, s) \frac{(\mathbf{k}\mathbf{q})^2}{q^2} d\mathbf{q} + \nu_m k^2 t \right] +$$

$$+ \int_0^t ds (t-s) \int f(p, s) B(q, 0) \frac{(\mathbf{k}\mathbf{p})^2 q^2 + k^2 (\mathbf{q}\mathbf{p})^2}{p^2 q^2} d\mathbf{p}. \quad (10)$$

From the asymptotic behavior of (10) at large t ($t \gg \tau$), one can construct an equation for $B(k, t)$:

$$\frac{\partial B(k, t)}{\partial t} + 2(\chi + \nu_m) k^2 B(k, t) = \pi \int I(p, 0) B(q, t) \frac{(\mathbf{k}\mathbf{p})^2 q^2 + k^2 (\mathbf{q}\mathbf{p})^2}{p^2 q^2} d\mathbf{p}; \quad (11)$$

$$\chi = \frac{\pi}{3} \int I(q, 0) d\mathbf{q} = \frac{\pi^2}{48c^3} \int k^2 f^2 \left(\frac{k}{2}, 0 \right) d\mathbf{k}.$$

(11) can be transformed to r -space:

$$\frac{\partial B_1}{\partial t} = 2 \left\{ AB_1'' + \frac{A_1}{r} B_1' - 2 \frac{v'}{r} B_1 \right\};$$

$$B_1 = r^{-3} \int_0^r \rho^2 B(\rho, t) d\rho; \quad B(r, t) = \int B(k, t) e^{i(\mathbf{k}, \mathbf{r})} d\mathbf{k};$$

$$v(r) = \pi \int I(k, 0) e^{i(\mathbf{k}, \mathbf{r})} d\mathbf{k}; \quad A = \chi + \nu_m - (rv_1' + v_1); \quad (12)$$

$$A_1 = 4(\chi + \nu_m) - (\nu' r + 2\nu_1' r + 4\nu_1); \quad \nu_1 = r^{-3} \int_0^r \nu(\rho) \rho^2 d\rho.$$

III. We shall seek the eigenfunctions (12) by the substitution $B_1 = h e^{-2Et}$; then, by the substitution

$$z = h_1 \exp \left[\frac{1}{2} \int_0^r \frac{d\rho A_1(\rho)}{\rho A(\rho)} \right]$$

we pass to the Schrödinger equation

$$z'' + m(E - U)z = 0, \quad (13)$$

where the “mass” is $m = 1/A$, and the potential is

$$U = 2\nu'/r + \frac{1}{2} A(A_1/rA)' + A_1^2/4r^2 A.$$

As $r \rightarrow 0$, $U \rightarrow 2\nu_m/r^2$; as $r \rightarrow \infty$, $U \rightarrow 2(\chi + \nu_m)/r^2$. The wave function of a bound state in the potential corresponds to an unstable (growing) solution in the present problem, the eigenenergy of the state being the growth increment with the opposite sign. A bound state appears if U has a potential well. The well arises if $\chi \gg \nu_m$; indeed, in this case, for r such that

$$\nu(0)/\nu_2(0) \gg r^2 \gg \nu_m/\nu_2(0), \quad \text{where} \quad \nu_2(r) = \int \pi I(q, 0) q^2 dq,$$

$$U \simeq \nu_2(0)/15.$$

The presence of a potential well is not a sufficient condition for the existence of eigenfunctions with $E < 0$ —the well may prove insufficiently deep; moreover, the eigenfunction must have the properties of a correlation function (its Fourier image $B(k, t)$ must be positive). To determine a sufficient condition, let us turn to equation (11). By the substitution $B = h e^{-2Et}$ we pass to the eigenvalue

problem for an integral equation. To find the minimal E , we formulate the variational principle

$$E = \left[(\chi + \nu_m) \int k^2 h^2 dk - \frac{\pi}{2} \int dk dp h(k) h(q) I(p, 0) \frac{(\mathbf{k}\mathbf{p})^2 q^2 + k^2 (\mathbf{q}\mathbf{p})^2}{q^2 p^2} \right] / \int h^2 dk; \quad (14)$$

$$\delta E = 0.$$

Assign to $h(k)$ the following form: for $k \leq k_1$,

$$h(k) = k_0^{-1/2} k_1^{-3} k^2 \exp(-k_1/2k_0),$$

where

$$\nu_2(0)/\nu(0) \ll k_1^2 \ll \nu_2(0)/\nu_m, \quad k_0^2 \simeq \nu_2(0)/\nu_m;$$

for $k \geq k_1$,

$$h(k) = k^{-1} k^{-1/2} \exp(-k/2k_0).$$

For $k^2 < \nu_2(0)/\nu(0)$ only a negligible part of the energy $h(k)$ is concentrated; therefore the right-hand side of (14) can be simplified:

$$\begin{aligned} E &= \left[\nu_m \int k^2 h^2 dk - \nu_2(0)/5 \int \left\{ \frac{2}{3} h^2 + 2h h' k + h'' h k^2 \right\} dk \right] / \int h^2 dk = \\ &= -\frac{7}{30} \nu_2(0) + 2k_0^2 \nu_m + \nu_2 \alpha(k_1/k_0), \end{aligned}$$

where $\alpha(k_1/k_0)$ is a small quantity of order k_1/k_0 . It is clear that if $k_0^2 \lesssim 7\nu_2(0)/60\nu_m$, then our $h(k)$ gives the functional (14) a negative value. Consequently, the eigenfunctions of problem (11) with $E < 0$ exist. Thus, acoustic turbulence is unstable with respect to perturbations of the magnetic field. It is not difficult to show that $\min U(r) = -\frac{2}{3}\nu_2(0)$; therefore the growth increment of the field is

$$\gamma \simeq \nu_2(0) = \frac{\pi}{24c^3} \int k^4 f^2 \left(\frac{k}{2}, 0 \right) dk. \quad (15)$$

We see that the most essential quantity determining the possibility of field generation is

$$\nu(0)/\nu_m = 3\chi/\nu_m \simeq (v^3/c^3)v\lambda/\nu_m = S = M^3\text{Re}_m, \quad (16)$$

(M is the Mach number, Re_m is the magnetic Reynolds number). For instability it is necessary that $S \gg 1$.

IV. The velocity field of acoustic turbulence is potential. The vortical component is generated in the same way as the magnetic one, since the equations for $\mathbf{H}$ and $\text{rot } \mathbf{v}$ are analogous and $\text{rot } \mathbf{v}(\mathbf{r}, 0)$ may be taken to be statistically independent of the potential component. Here the quantity most essential for generation is $S_\omega = M^3\text{Re}$. If $S_\omega \gg 1$, the generation of vortices is analogous to the instability described above; if $S_\omega \ll 1$, there occurs growth of the vortical component v_ω as a nonlinear effect—a phenomenon known as acoustic streaming⁽²⁾.

The growth of v_ω for $S_\omega \gg 1$ is limited by turbulent viscosity: the point is that the vortex-vortex interaction has the usual character of energy transfer into the region of large k . For a rough estimate of the energy of the established level v_ω , we note that, since the width of the potential pit is $\sim \lambda$, the energy v_ω^2 is concentrated at $k \sim 1/\lambda$. Let us write the equation for v_ω :

$$\frac{d}{dt}v_\omega^2 = \gamma v_\omega^2 - v_\omega^2/\lambda.$$

In the stationary state $v_\omega \sim \lambda\gamma$. Let us calculate v_ω for the following form of E : for $k < 1/\lambda$, $E = 0$; for $1/\lambda \leq k \leq k_\nu$, $E = v^2\lambda^{1-\alpha}k^{-\alpha}$, where k_ν is the spectral cutoff due to viscosity; for $k > k_\nu$, $E = 0$. Here there are 3 cases: 1) $\alpha > 3/2$, then $\gamma \sim M^3v/\lambda$ and $v_\omega \sim M^3v$; 2) $1 < \alpha < 3/2$, then $\gamma \sim M^3(\lambda k_\nu)^{2-2\alpha}vk_\nu$, $k_\nu = \text{Re}^{2/(1+\alpha)}\lambda^{-1}$, $v_\omega \sim vM^3\text{Re}^{2(3-2\alpha)/(1+\alpha)} \leq vS_\omega$ (for $\alpha < 3/2$, $v_\omega \sim M^3v$); 3) $\alpha = 3/2$, then $\gamma \sim M^3 \ln(k_\nu\lambda)v/\lambda$, $v_\omega \sim M^3v \ln \text{Re}$.

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Note: Figure translations are in progress. See original paper for figures.

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