

# ON THE PARTIAL ORDERABILITY OF RINGS AND ALGEBRAS

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**Abstract**

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*MATHEMATICS*

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## ON THE PARTIAL ORDERABILITY OF RINGS AND ALGEBRAS

*(Presented by Academician L. V. Kantorovich on 17 VI 1969)*

1. The question of the structural ordering of rings (algebras) without nilpotent elements, i.e., of turning them into  $f$ -rings ( $f$ -algebras) in the sense of G. Birkhoff and R. Pierce <sup>(1)</sup>, was considered in a previous paper by the author <sup>(2)</sup>. Far from always, however, is such an ordering possible. Nevertheless, it is often possible to introduce in a ring (algebra)  $X$  a sufficiently good partial order, well coordinated with the operations present in  $X$ . Here we study the question of the possibility (and also uniqueness) of introducing such partial orders in  $X$ .
2. By a **ring**  $X$  we shall everywhere mean a commutative associative ring without nonzero nilpotents, whose additive group has no torsion. By a **cone** in  $X$  we mean any additive semigroup  $K$  containing 0, closed with respect to multiplication (i.e., for  $K$  one has  $K \cdot K \subset K$ ), isolated (i.e., such that if  $nx \in K$  for some  $n \in N$ , where  $N$  is the set of natural numbers, then also  $x \in K$ ), and not containing simultaneously the elements  $x$  and  $-x$  for  $x \neq 0$ . A **partially ordered** (p.o.) **ring** is a ring  $X$  with a cone of positive elements  $X^+$ ; here  $x \geq y$  means that  $x - y \in X^+$ .

**Definition.** By a **partially ordered ring of functions** we shall mean a ring which is a subdirect product of (totally) ordered integral domains.

G. Tärper <sup>(12)</sup> showed that a ring  $X$  can be turned into a p.o. ring of functions if and only if no nontrivial sums of squares are annihilated in  $X$ ; in other words, if from

$$x_1^2 + \dots + x_n^2 = 0$$

it follows that

$$x_1 = \dots = x_n = 0.$$

A. Hayes <sup>(13)</sup> found conditions under which a p.o. ring is isomorphic to a p.o. ring of functions. These conditions are as follows:

- 1)  $\forall x \in X (x^2 \in X^+)$ ;    2)  $\forall x \in X^+ \forall y \in X^+ \forall n \in N (xy - x^{2n} \in X^+)$ .

He called such a ring an  $f^*$ -ring. A special case of an  $f^*$ -ring is an  $f$ -ring (commutative and containing no nonzero nilpotents; in what follows, by an  $f$ -ring we shall, without any reservation, understand only such an  $f$ -ring). We shall call the cone of positive elements of an  $f^*$ -ring (of an  $f$ -ring) an  $f^*$ -**cone** (respectively, an  $f$ -**cone**). We are interested in the question of turning a ring into a p.o. ring more “qualified” than simply an  $f^*$ -ring.

3. The annihilator of a set  $G$  in a ring  $X$  will be denoted by  $X(G)$ , and for  $G = \{x\}$  by  $X(x)$ . Instead of  $X(X(x))$  we shall write  $X_x$ . The binary relation  $xy = 0$  in  $X$  is a disjointness relation <sup>(4)</sup>. The disjoint complements with respect to this relation are precisely the annihilators in  $X$ . This circumstance justifies our use of the term **component** instead of the term annihilator. We also note that if  $X$  is an  $f$ -ring, then the set of components in the sense of the theory of semiordered spaces of L. V. Kantorovich (see, for example, <sup>(7)</sup>) in the  $l$ -group  $X$  coincides with the set of annihilators in the ring  $X$ . A component of the form  $X_x$  will be called **principal**.

If  $x = yz$  and  $X(x) = X(z)$ , then the element  $z$  is called a **partial**  $x/y$ .

Partially defined not for all pairs  $(x, y)$ , but there may exist only one for a given pair.

**Definition.** A cone in a ring  $X$  is called a  $u$ -cone if it contains the squares (and hence also their sums) of all elements of  $X$  and is closed with respect to partial division.

Let us note at once that  $u$ -cones can exist only in rings in which nontrivial sums of squares are not annihilated. Obviously, every  $f^*$ -cone is a  $u$ -cone.

By the **projection** of an element  $x$  onto a component  $X'$  we mean an  $x' \in X'$  (if it exists) such that

$$(x - x') \cdot X' = \{0\}.$$

Projections of an element will also be called its **fragments**. It is not hard to show that every  $u$ -cone is closed with respect to the operation of taking a fragment.

**Definition.** We shall call a  $u$ -cone  $K$  in a ring  $X$  a  $\varphi$ -cone if the following conditions A and B are fulfilled:

- A.  $\forall x \in X \forall G \subset X [\{xg\}(g \in G) \subset K] \& [X(G) = X(x)] \Rightarrow x \in K.$
- B.  $\forall x \in X \exists G \subset X [\{xg\}(g \in G) \subset K \cup (-K)] \& [X(G) = X(x)].$

A partially ordered ring  $X$  with a  $\varphi$ -cone  $X^+$  will be called a  $\varphi$ -ring.

**Theorem 1.** *The set of all  $\varphi$ -cones in the ring  $X$  coincides with the set of all maximal  $u$ -cones and with the set of all maximal  $f^*$ -cones.*

**Theorem 2.** *A ring in which nontrivial sums of squares are not annihilated can be turned into a  $\varphi$ -ring.*

This theorem is a nontrivial strengthening of the cited result of G. Thierrin. Indeed, not every ring of functions satisfies conditions A and B. Condition A for a partially ordered ring of functions (and a  $\varphi$ -ring, being an  $f^*$ -ring, can be represented in this way) means that a function “glued together” from any (and not only finite) set of positive “pieces” (in particular, from a set of positive functions) is itself positive. B means that any function can be “glued together” from positive and negative “pieces.”

Obviously, every  $f$ -cone is a  $\varphi$ -cone, but the converse, of course, is not true. In a  $\varphi$ -cone  $K \subset X$  the condition

$$B. \quad \forall x_1 \in K \dots \forall x_n \in K \quad (x_1 \dots x_n = 0) \Rightarrow \inf\{x_i\} = 0.$$

holds. In an  $f$ -cone condition  $\Gamma$ , the converse to B, is also fulfilled.

**Definition.** A  $\varphi$ -cone satisfying in addition condition  $\Gamma$  will be called an  $f$ -cone. A partially ordered ring  $X$  with an  $f$ -cone  $X^+$  will be called an  $f$ -ring.

Although every  $\varphi$ -ring admits a representation in the form of a partially ordered ring of functions, only for an  $f$ -ring is such a representation possible in which the lattice operations (for finite sets) are computed coordinatewise.

**Definition.** A ring  $X$  is called a ring with projections (with principal projections) if in  $X$  there exists the projection of any element onto any component (onto any principal component), i.e. every component (respectively every principal component) is a direct factor.

If one restricts oneself to the class of rings considered by us, then, under the additional requirement of the existence of a multiplicative identity, rings with projections turn out to be exactly Baer rings, and rings with principal projections Rickart rings (see, for example, (14)) in this class of rings.

**Theorem 3.** *A  $\varphi$ -cone in a ring with principal projections is an  $f$ -cone.*

**Theorem 4.** *A ring with principal projections in which nontrivial sums of squares are not annihilated can be turned into an  $f$ -ring.*

**Theorem 5.** *A  $\varphi$ -cone in a ring with projections is an  $f$ -cone.*

From this the following result follows immediately from (3).

**Theorem 6.** *A ring with projections in which nontrivial sums of squares do not annihilate can be made into an  $f$ -ring.*

4. In (3) it was shown that every ring can be embedded in a sufficiently good way in a ring with projections.

**Definition.** A subring  $X$  of a ring  $Y$  is called **pseudodense** in  $Y$  if for every  $0 \neq y \in Y$ , in the component  $Y_y$  there is a  $0 \neq x \in X$ . The ring  $\overline{X}$ , the smallest of all rings with projections containing the given ring  $X$  as a pseudodense subring (this means that if a ring  $Y$  contains  $X$  in a pseudodense way, then  $\overline{X}$  can be embedded isomorphically in  $Y$ , leaving the elements of  $X$  fixed), is called the

**completion of the ring  $X$  by means of projections**, or the  $P$ -completion of  $X$ .

The  $P$ -completion of the ring  $X$  exists and is unique in the sense that all  $P$ -completions of  $X$  are isomorphic under an isomorphism leaving the elements of  $X$  fixed. The construction of  $\bar{X}$  was given in (3). Without presenting it here, we note only that every element of  $\bar{X}$  is a sum of fragments in  $\bar{X}$  of elements of  $X$ . We also note that in (3) the definition of the  $P$ -completion was inaccurate: the  $P$ -completion of  $X$  was defined there simply as the smallest of the rings with projections containing  $X$ ; however, in principle such a smallest ring may not exist.

**Theorem 7.** If  $X$  is a  $\varphi$ -ring and  $\bar{X}$  is the  $P$ -completion of the ring  $X$ , then the partial order on  $X$  can be extended to  $\bar{X}$ , making the latter an  $f$ -ring. Moreover, if  $X$  is an  $f$ -ring, then one can additionally arrange that  $X$  be embedded in the  $f$ -ring  $\bar{X}$  with preservation of suprema of finite sets.

**Corollary.** A ring in which nontrivial sums of squares do not annihilate is isomorphic to a subring of some  $f$ -ring.

**5.** Let us touch on the question of when a partial order with one or another property can be introduced into a ring in exactly one way.

In any ring  $X$  in which nontrivial sums of squares do not annihilate, there exist  $u$ -cones. Their intersection is also a  $u$ -cone; let  $K_0$  be, obviously, the smallest one in  $X$ . The cone  $K_0$  is easily described for a ring with principal projections.

**Theorem 8.** In order that a ring  $X$  be made into a  $\varphi$ -ring in exactly one way, it is necessary and sufficient that in  $X$  nontrivial sums of squares do not annihilate and that for every  $x \neq 0$  there be a  $0 \neq u \in K_0$  such that  $0 \neq xu \in K_0 \cup (-K_0)$ .

**Theorem 9.** In order that a ring  $X$  with principal projections be made into an  $f$ -ring (and into a  $\varphi$ -ring) in exactly one way, it is necessary and sufficient that in  $X$  nontrivial sums of squares do not annihilate and that every nonzero element have a nonzero fragment of the form  $\pm \sum x_i^2 / \sum y_i^2$ .

**Theorem 10** (see (3)). In order that a ring  $X$  with projections be made into an  $f$ -ring (and into a  $\varphi$ -ring) in exactly one way, it is necessary and sufficient that the conditions of the preceding theorem hold for it.

**6.** Everything said above for the ring  $X$  carries over, without special changes, to the case of an algebra  $X$ . Here by an **algebra** is meant a ring (commutative, associative, and without nonzero nilpotents) which is an algebra over the field  $R$  of real numbers. In this case one need only require of the cone  $K$  that  $x \in K, \lambda \in R^+$  imply  $\lambda x \in R^+$ .

A particular case of the problem considered in § 5 is the following. When in a given  $f$ -algebra ( $f$ -ring)  $X$  there is no second  $f$ -cone, i.e. there is no second structural order compatible in the proper way with the operations in the algebra (ring)  $X$ . In the case when there is no second structural order, we shall say that in  $X$  multiplication ...

defines a **structural order** (or an  **$f$ -order**). If in the given  $\varphi$ -algebra ( $\varphi$ -ring)  $X$  there is no second  $\varphi$ -cone, we shall say that in  $X$  multiplication defines a  $\varphi$ -order.

**Definition.** A  $K$ -algebra ( $K_\sigma$ -algebra) is called an  $f$ -algebra if it is simultaneously a  $K$ -space ( $K_\sigma$ -space) (see (7)).

From Theorem 9 there follows the following result, the first half of which was obtained in (2).

**Theorem 11.** *In a  $K_\sigma$ -algebra, multiplication defines a structural order and even a  $\varphi$ -order.*

Let us illustrate this theorem by an example. In functional analysis one often considers (commutative) partially ordered algebras of bounded self-adjoint operators in a Hilbert space  $H$ . A number of works (<sup>1, 5, 6, 8-11</sup>) have been devoted to the study of such partially ordered algebras. These are algebras of operators partially ordered in the natural way: an operator  $A \geq 0$  means that  $(Ah, h) \geq 0$  ( $h \in H$ ). This **natural partial order** usually turns the corresponding algebra of operators into an  $f$ -algebra. This is the case, for example, for a strongly closed (in the sense of operator theory) algebra, or for an algebra of the so-called type  $S''$ . By a **semiordered algebra of self-adjoint operators** we shall mean an algebra of operators that is a  $K$ -algebra with respect to the natural partial order. From Theorem 11 the following proposition follows.

**Corollary.** *In a semiordered algebra  $X$  of self-adjoint operators, multiplication defines a structural order (and even a  $\varphi$ -order). In other words, every structural order in the algebra  $X$  for which the conditions*

$$1) A \geq 0, B \geq 0 \Rightarrow AB \geq 0; \quad 2) A^2 \wedge B^2 = 0 \iff AB = 0,$$

*are satisfied, coincides with the natural one.*

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