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Abstract

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THEORY OF ELASTICITY

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A THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY FOR A DOMAIN BOUNDED BY TWO PARALLEL PLANES

(Presented by Academician A. Yu. Ishlinskii, 22 V 1969)

The problem of the equilibrium of an elastic body bounded by the planes $z = \pm h/2$ and subjected to surface forces $\mathbf{q}^+(x, y)$ and $\mathbf{q}^-(x, y)$, acting on the boundary surfaces, was reduced by V. Z. Vlasov in [1] to the solution of a system of differential equations of infinite order. In the present article we consider a more general problem in which, in addition to the surface forces $\mathbf{q}^+$ and $\mathbf{q}^-$, arbitrary body forces $\mathbf{Q}(x, y, z)$ act, and we indicate an explicit solution of this problem in the form of series whose construction is based on expansions in the eigenfunctions of a certain non-self-adjoint differential operator.

Let us first note that by the substitution

$$\begin{aligned} u_x &= u_0(x, y) + u_1(x, y)z + u(x, y, z), \\ u_y &= v_0(x, y) + v_1(x, y)z + v(x, y, z), \\ u_z &= w_0(x, y) + w_1(x, y)z + w_2(x, y)z^2 + w(x, y, z) \end{aligned} \quad (1)$$

one can reduce the problem under consideration to the case in which, instead of the given system of forces $\mathbf{Q}$, $\mathbf{q}^+$, and $\mathbf{q}^-$, the elastic body is acted upon by a system of forces $\mathbf{P}$, $\mathbf{p}^+$, and $\mathbf{p}^-$ satisfying the conditions

$$\begin{aligned} \int_{-h/2}^{h/2} \mathbf{P} dz + \mathbf{p}^+ + \mathbf{p}^- &= 0, \quad \int_{-h/2}^{h/2} \mathbf{P} z dz + \frac{h}{2} (\mathbf{p}^+ - \mathbf{p}^-) = 0, \\ \int_{-h/2}^{h/2} P_z \left(\frac{h^2}{4} - z^2 \right) dz &= 0. \end{aligned} \quad (2)$$

Substituting the elastic displacements u_x , u_y , and u_z , expressed by formulas (1), into the equilibrium equations of the elastic body, we obtain from conditions (2) the equations

$$G \left[\nabla^2 u_0 + \frac{1}{1-2\nu} \left(\frac{\partial \Delta_0}{\partial x} + 2\nu \frac{\partial w_1}{\partial x} \right) \right] + X_0 = 0, \quad G \left[\nabla^2 v_0 + \frac{1}{1-2\nu} \left(\frac{\partial \Delta_0}{\partial y} + 2\nu \frac{\partial w_1}{\partial y} \right) \right] + Y_0 = 0, \quad G \left\{ \frac{h^2}{12} \nabla^2 w_2 + \nabla^2 w_0 + \Delta_1 \right\} + Z_0 = 0,$$

for the functions u_0 , v_0 , w_1 , and the equations

$$G \left(\frac{h^2}{12} \nabla^2 w_2 + \nabla^2 w_0 + \Delta_1 \right) + Z_0 = 0,$$

$$G \left\{ \frac{h^2}{12} \left[\nabla^2 u_1 + \frac{1}{1-2\nu} \left[\frac{\partial \Delta_1}{\partial x} - (1-6\nu) \frac{\partial w_2}{\partial x} \right] \right] - u_1 - \frac{\partial w_0}{\partial x} \right\} + X_1 = 0,$$

$$G \left\{ \frac{h^2}{12} \left[\nabla^2 v_1 + \frac{1}{1-2\nu} \left[\frac{\partial \Delta_1}{\partial y} - (1-6\nu) \frac{\partial w_2}{\partial y} \right] \right] - v_1 - \frac{\partial w_0}{\partial y} \right\} + Y_1 = 0,$$

$$G \left\{ \frac{h^2}{120} \nabla^2 w_2 + \frac{1}{6} \left[\nabla^2 w_0 + \frac{1}{1-2\nu} [\Delta_1 + 4(1-\nu)w_2] \right] \right\} + Z_2 = 0 \quad (4)$$

for the functions w_0 , u_1 , v_1 , w_2 , where G is the shear modulus, ν is Poisson's ratio, and

$$\Delta_0 = \partial u_0 / \partial x + \partial v_0 / \partial y, \quad \Delta_1 = \partial u_1 / \partial x + \partial v_1 / \partial y,$$

$$X_0 = \frac{1}{h} \left(\int_{-h/2}^{h/2} Q_x dz + q_x^+ + q_x^- \right), \quad Y_0 = \frac{1}{h} \left(\int_{-h/2}^{h/2} Q_y dz + q_y^+ + q_y^- \right),$$

$$Z_0 = \frac{1}{h} \left(\int_{-h/2}^{h/2} Q_z dz + q_z^+ + q_z^- \right), \quad X_1 = \frac{1}{h} \left[\int_{-h/2}^{h/2} Q_x z dz + \frac{h}{2} (q_x^+ - q_x^-) \right], \quad (5)$$

$$Y_1 = \frac{1}{h} \left[\int_{-h/2}^{h/2} Q_y z dz + \frac{h}{2} (q_y^+ - q_y^-) \right],$$

$$Z_1 = \frac{1}{h} \left[\int_{-h/2}^{h/2} Q_z z dz + \frac{h}{2} (q_z^+ - q_z^-) \right], \quad Z_2 = \frac{10}{h^3} \int_{-h/2}^{h/2} Q_z \left(\frac{h^2}{4} - z^2 \right) dz.$$

To find the elastic displacements u, v, w appearing in formulas (1), we introduce for consideration the one-dimensional non-self-adjoint boundary-value problem

$$\begin{aligned} \nu d^2 f / d\xi^2 - \frac{1}{2} d^2 g / d\xi^2 &= \lambda^2 (1 - \nu) f, \\ 2d^2 f / d\xi^2 - (2 - \nu) d^2 g / d\xi^2 &= \lambda^2 (1 - \nu) g, \end{aligned} \quad (6)$$

$$-\frac{1}{2} < \xi < \frac{1}{2}, \quad g = dg/d\xi = 0 \quad \text{for } \xi = \pm \frac{1}{2}.$$

The sequence of eigenfunctions of this boundary-value problem consists of the functions $f_0^{(1)} = 1$, $g_0^{(1)} = 0$ and $f_0^{(2)} = \xi$, $g_0^{(2)} = 0$, corresponding to the eigenvalue $\lambda = 0$, and of two sequences $f_k^{(1)}(\xi), g_k^{(1)}(\xi)$ and $f_k^{(2)}(\xi), g_k^{(2)}(\xi)$, corresponding to the eigenvalues $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$, $k = 1, 2, \dots$, where $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ are the nonzero roots of the equations

$$\lambda + \sin \lambda = 0, \quad \lambda - \sin \lambda = 0, \quad (7)$$

lying in the right half-plane (the roots of equations (7) lying in the left half-plane do not give new eigenfunctions). By $f_k^{(1)*}(\xi), g_k^{(1)*}(\xi)$ and $f_k^{(2)*}(\xi), g_k^{(2)*}(\xi)$, $k = 0, 1, 2, \dots$, we shall denote the eigenfunctions of the boundary-value problem

$$\begin{aligned} \nu d^2 f^* / d\xi^2 + 2d^2 g^* / d\xi^2 &= \lambda^2 (1 - \nu) f^*, \\ -\frac{1}{2} d^2 f^* / d\xi^2 - (2 - \nu) d^2 g^* / d\xi^2 &= \lambda^2 (1 - \nu) g^*, \end{aligned} \quad (8)$$

$$-\frac{1}{2} < \xi < \frac{1}{2}, \quad \nu f^* + 2g^* = \nu df^*/d\xi + 2dg^*/d\xi = 0 \quad \text{for } \xi = \pm \frac{1}{2},$$

adjoint to the boundary-value problem (6), normalized in such a way that, along with the biorthogonality conditions

$$\int_{-1/2}^{1/2} (f_k^{(1)} f_l^{(2)*} + g_k^{(1)} g_l^{(2)*}) d\xi = \int_{-1/2}^{1/2} (f_k^{(2)} f_l^{(1)*} + g_k^{(2)} g_l^{(1)*}) d\xi = 0, \quad k \geq 0, l \geq 0, \quad (9)$$

$$\int_{-1/2}^{1/2} (f_k^{(1)} f_l^{(1)*} + g_k^{(1)} g_l^{(1)*}) d\xi = \int_{-1/2}^{1/2} (f_k^{(2)} f_l^{(2)*} + g_k^{(2)} g_l^{(2)*}) d\xi = 0, \quad k \neq l,$$

the conditions

$$\int_{-1/2}^{1/2} (f_k^{(1)} f_k^{(1)*} + g_k^{(1)} g_k^{(1)*}) d\xi = \int_{-1/2}^{1/2} (f_k^{(2)} f_k^{(2)*} + g_k^{(2)} g_k^{(2)*}) d\xi = 1. \quad (10)$$

A pair of functions $F(\xi)$ and $G(\xi)$, given on the interval $-\frac{1}{2} \leq \xi \leq \frac{1}{2}$, can be represented by the series

$$F(\xi) = c_0^{(1)} + c_0^{(2)} \xi + \sum_{k=1}^{\infty} [c_k^{(1)} f_k^{(1)}(\xi) + c_k^{(2)} f_k^{(2)}(\xi)], \quad (11)$$

$$G(\xi) = \sum_{k=1}^{\infty} [c_k^{(1)} g_k^{(1)}(\xi) + c_k^{(2)} g_k^{(2)}(\xi)].$$

where

$$c_0^{(1)} = \int_{-1/2}^{1/2} [F(\zeta) - \frac{\nu}{2} G(\zeta)] d\zeta, \quad c_0^{(2)} = 12 \int_{-1/2}^{1/2} [F(\zeta) - \frac{\nu}{2} G(\zeta)] \zeta d\zeta,$$

$$c_k^{(j)} = \int_{-1/2}^{1/2} [F(\zeta) f_k^{(j)*}(\zeta) + G(\zeta) g_k^{(j)*}(\zeta)] d\zeta, \quad j = 1, 2, \quad k = 1, 2, \dots \quad (12)$$

The completeness of the biorthogonal system of eigenfunctions constructed above and the convergence of the corresponding expansions (11) follow from known results in the theory of linear differential operators (see (2)).

Instead of the prescribed system of surface forces $\mathbf{p}^+$, $\mathbf{p}^-$ and body forces $\mathbf{P}$, we shall consider below the equivalent body forces $\mathbf{P}^*$, where

$$\mathbf{P}^* = \mathbf{P} + \delta(z, h/2) \mathbf{p}^+ + \delta(z, -h/2) \mathbf{p}^- \quad (13)$$

$$\left(\delta(z, z_0) = 0 \text{ for } z \neq z_0, \quad \int_{-h/2}^{h/2} \delta(z, z_0) dz = 1 \right).$$

Putting successively in (11) and (12) $F = P_x^*(x, y, h\xi)$, $G = 0$; $F = P_y^*(x, y, h\xi)$, $G = 0$ and $F = 0$, $G = \int_{-1/2}^{\xi} P_z^*(x, y, h\xi') d\xi'$, and taking into account the equalities (2), we obtain the expansions

$$P_x^*(x, y, z) = \sum_{j=1}^2 \sum_{k=1}^{\infty} a_k^{(j)}(x, y) f_k^{(j)}\left(\frac{z}{h}\right), \quad P_y^*(x, y, z) = \sum_{j=1}^2 \sum_{k=1}^{\infty} b_k^{(j)}(x, y) f_k^{(j)}\left(\frac{z}{h}\right),$$

$$P_z^*(x, y, z) = \sum_{j=1}^2 \sum_{k=1}^{\infty} c_k^{(j)}(x, y) \varphi_k^{(j)}\left(\frac{z}{h}\right), \quad (14)$$

where

$$a_k^{(j)}(x, y) = \frac{1}{h} \left[\int_{-h/2}^{h/2} P_x(x, y, z') f_k^{(j)*}\left(\frac{z'}{h}\right) dz' + p_x^+(x, y) f_k^{(j)*}\left(\frac{1}{2}\right) + p_x^-(x, y) f_k^{(j)*}\left(-\frac{1}{2}\right) \right],$$

$$b_k^{(j)}(x, y) = \frac{1}{h} \left[\int_{-h/2}^{h/2} P_y(x, y, z') f_k^{(j)*}\left(\frac{z'}{h}\right) dz' + p_y^+(x, y) f_k^{(j)*}\left(\frac{1}{2}\right) + p_y^-(x, y) f_k^{(j)*}\left(-\frac{1}{2}\right) \right],$$

$$c_k^{(j)}(x, y) = -\frac{1}{h} \left[\int_{-h/2}^{h/2} P_z(x, y, z') \psi_k^{(j)}\left(\frac{z'}{h}\right) dz' + p_z^+(x, y) \psi_k^{(j)}\left(\frac{1}{2}\right) + p_z^-(x, y) \psi_k^{(j)}\left(-\frac{1}{2}\right) \right], \quad (15)$$

$$\varphi_k^{(j)} = \frac{dg_k^{(j)}}{d\xi}, \quad \psi_k^{(j)} = \int_0^{\xi} g_k^{(j)}(\xi') d\xi'. \quad (16)$$

In this case the identical equality will hold

$$\sum_{j=1}^2 \sum_{k=1}^{\infty} c_k^{(j)}(x, y) f_k^{(j)}\left(\frac{z}{h}\right) = 0. \quad (17)$$

Let us now consider the elastic displacements defined by the formulas

$$u = \frac{h}{2\pi G} \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial x} f_k^{(j)} \left(\frac{z}{h} \right), \quad v = \frac{h}{2\pi G} \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial y} f_k^{(j)} \left(\frac{z}{h} \right),$$

$$w = \frac{h}{2\pi G} \sum_{j=1}^2 \sum_{k=1}^{\infty} \Phi_k^{(j)} \frac{d}{dz} \left[g_k^{(j)} \left(\frac{z}{h} \right) - f_k^{(j)} \left(\frac{z}{h} \right) \right], \quad (18)$$

where

$$\begin{aligned} \Phi_k^{(j)}(x, y, x', y') &= a_k^{(j)}(x', y') U_k^{(j)}(x - x', y - y') + \\ &+ b_k^{(j)}(x', y') V_k^{(j)}(x - x', y - y') + c_k^{(j)}(x', y') W_k^{(j)}(x - x', y - y'), \quad (19) \\ U_k^{(j)}(\xi, \eta) &= -\frac{1}{2h} \int_{\xi}^{\infty} W_k^{(j)}(\xi', \eta) d\xi' \quad \text{for } \xi > 0, \\ U_k^{(j)}(-\xi, \eta) &= -U_k^{(j)}(\xi, \eta), \quad V_k^{(j)}(\xi, \eta) = U_k^{(j)}(\eta, \xi), \\ W_k^{(j)}(\xi, \eta) &= K_0 \left(\frac{\lambda_k^{(j)}}{h} \sqrt{\xi^2 + \eta^2} \right) \end{aligned} \quad (20)$$

($K_0(z)$ is the Macdonald function of zero order). According to (6) and (17), in the region $|z| \leq h/2$, $r \geq \varepsilon$ ($r = \sqrt{(x - x')^2 + (y - y')^2}$), the elastic displacements (18) correspond to body forces identically equal to zero, and to surface forces $\mathbf{p}$ equal to zero on the planes $z = \pm h/2$ and satisfying the limiting relations

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} 2\pi \varepsilon p_x &= \sum_{j=1}^2 \sum_{k=1}^{\infty} a_k^{(j)}(x', y') f_k^{(j)} \left(\frac{z}{h} \right), \\ \lim_{\varepsilon \rightarrow 0} 2\pi \varepsilon p_y &= \sum_{j=1}^2 \sum_{k=1}^{\infty} b_k^{(j)}(x', y') f_k^{(j)} \left(\frac{z}{h} \right), \quad (21) \\ \lim_{\varepsilon \rightarrow 0} 2\pi \varepsilon p_z &= \sum_{j=1}^2 \sum_{k=1}^{\infty} c_k^{(j)}(x', y') \varphi_k^{(j)} \left(\frac{z}{h} \right) \end{aligned}$$

on the cylindrical boundary surface $r = \varepsilon$. In accordance with the expansions (14), formulas (18) determine the elastic displacements caused by body forces $\mathbf{P}^*(x', y', z) \delta(x - x', y - y')$ ($\delta(\xi, \eta) = 0$ for $\xi^2 + \eta^2 \neq 0$, $\iint \delta(\xi, \eta) d\xi d\eta = 1$).

Thus, the elastic displacements caused by body forces $\mathbf{P}^*(x, y, z)$ acting in the region $x, y \in S$, $|z| \leq h/2$, can be found from the formulas

$$\begin{aligned}
 u(x, y, z) &= \frac{h}{2\pi G} \iint_S \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial x} f_k^{(j)}\left(\frac{z}{h}\right) dx' dy', \\
 v(x, y, z) &= \frac{h}{2\pi G} \iint_S \sum_{j=1}^2 \sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial y} f_k^{(j)}\left(\frac{z}{h}\right) dx' dy', \quad (22) \\
 w(x, y, z) &= \frac{h}{2\pi G} \iint_S \sum_{j=1}^2 \sum_{k=1}^{\infty} \Phi_k^{(j)} \frac{d}{dz} \left[g_k^{(j)}\left(\frac{z}{h}\right) - f_k^{(j)}\left(\frac{z}{h}\right) \right] dx' dy'.
 \end{aligned}$$

As the distance from the region in which the body forces $\mathbf{P}^*$ act increases, the elastic displacements u, v, w rapidly decrease in accordance with the exponential law of decrease of the Macdonald function $K_0(z)$ as $\text{Re } z \rightarrow +\infty$.

The solution indicated by us of the three-dimensional problem of the theory of elasticity for a region bounded by the planes $z = \pm h/2$ opens the possibility of a correct calculation of local deformations in elastic plates. The method applied above of expansion in eigenfunctions of the boundary-value problem (6) can also be used for a refined calculation of edge effects arising in an elastic plate of a prescribed configuration.

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Note: Figure translations are in progress. See original paper for figures.

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