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Abstract

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MATHEMATICS

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ON ANALYTIC TRANSFORMATIONS OF A NORMAL VECTOR

(Presented by Academician Yu. V. Linnik on 15 VI 1970)

Let $X = (x_1, \dots, x_n)$ be a random vector whose components x_j , $1 \leq j \leq n$, are independent random variables distributed according to the normal law $N(0, 1)$. Yu. V. Linnik posed the problem of describing the class Ξ of entire functions $\varphi(Z)$, $Z = (z_1, \dots, z_n) \in C^n$, real for $Z = X \in R^n$, and such that the random variable $y = \varphi(X)$ is distributed according to the law $N(0, 1)$.

In measure-theoretic terms, the class Ξ can be defined as the class of entire functions $\varphi(Z)$, $Z \in C^n$, real for $Z = X \in R^n$, and such that for every Borel set $B \subset R^1$ the equality

$$\frac{1}{\sqrt{2\pi}} \int_B e^{-y^2/2} dy = \left(\frac{1}{\sqrt{2\pi}} \right)^n \int_{\varphi^{-1}(B)} e^{-|X|^2/2} dX, \quad (1)$$

holds, where $\varphi^{-1}(B) = \{X : X \in R^n, \varphi(X) \in B\}$ is the full preimage of the set B in R^n , and dX is the volume element in R^n .

As is known, the class Ξ is nonempty; the general form of a linear function $\varphi(Z)$ of class Ξ is as follows:

$$\varphi(Z) = \sum_{k=1}^n a_k z_k, \quad (a_1, \dots, a_n) \in R^n, \quad \sum_{k=1}^n a_k^2 = 1. \quad (2)$$

However, examples can also be given of nonlinear functions of the class Ξ (Yu. V. Linnik):

$$\varphi(z_1, z_2) = z_1 \cos[\eta(z_1^2 + z_2^2)] + z_2 \sin[\eta(z_1^2 + z_2^2)], \quad (3)$$

$$\varphi(z_1, z_2, z_3) = z_1 \cos[\eta(z_3)] + z_2 \sin[\eta(z_3)],$$

where $\eta(z)$ is an arbitrary nonconstant entire function, real for $z \in R^1$.

For every entire function $\varphi(Z)$, $Z \in C^n$, put

$$M(r, \varphi) = \max_{Z \in C^n, |Z| \leq r} |\varphi(Z)|, \quad \left(|Z|^2 = \sum_{k=1}^n |z_k|^2 \right).$$

Yu. V. Linnik and V. L. Eidlin ⁽¹⁾ showed that if $\varphi(Z) \in \Xi$ and $\ln M(r, \varphi) = O(\ln^2 r)$, then the function $\varphi(Z)$ has the form (2). In the present paper we strengthen this result.

Theorem 1. If $\varphi(Z) \in \Xi$, then either $\varphi(Z)$ is a linear function of the form (2), or

$$\liminf_{r \rightarrow \infty} r^{-1} \ln M(r, \varphi) > 0. \quad (4)$$

For the function (3) with $\eta(z) = kz$ one has

$$\lim_{r \rightarrow \infty} r^{-1} \ln M(r, \varphi) = |k|,$$

and therefore the assertion of Theorem 1 is, in a certain sense, unimprovable.

Theorem 1 follows directly from the two following lemmas.

Lemma 1. If $\varphi(Z) \in \Xi$, then

$$\int_{|X|>1} |\varphi(X)| |X|^{-n-2} dX < \infty \quad (X \in R^n). \quad (5)$$

Lemma 2. If an entire function $\varphi(Z)$ does not satisfy condition (4), but does satisfy condition (5), then it is linear.

For the proof of Lemma 1, denote by $E_{k,N}$ the set

$$E_{k,N} = \{X : X \in R^n, |X| < N, \sqrt{k}N \leq |\varphi(X)| < \sqrt{k+1}N\},$$

$$(N > 1, k = 0, 1, 2, \dots).$$

Let $\text{mes}_n E_{k,N}$ be the Lebesgue measure of this set in R^n . Obviously, $\text{mes}_n E_{k,N} < (2N)^n$. We shall show that for $k \geq 1$ the following sharper estimate is valid:

$$\text{mes}_n E_{k,N} \leq 2(\sqrt{2\pi})^{n-1} (\sqrt{k}N)^{-1} \exp\{-(k-1)N^2/2\}. \quad (6)$$

For this, put in (1)

$$B = \{y : y \in R^1, |y| > \sqrt{k}N\}.$$

We have

$$\frac{1}{\sqrt{2\pi}} \int_B e^{-y^2/2} dy = \frac{2}{\sqrt{2\pi}} \int_{\sqrt{k}N}^{\infty} e^{-y^2/2} dy < \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{k}N} e^{-kN^2/2}, \quad (7)$$

and, on the other hand,

$$\begin{aligned} \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\varphi^{-1}(B)} e^{-|X|^2/2} dX &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{|\varphi(X)| > \sqrt{k}N} e^{-|X|^2/2} dX \geq \\ &\geq \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{E_{k,N}} e^{-|X|^2/2} dX \geq \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-N^2/2} \text{mes}_n E_{k,N}. \end{aligned} \quad (8)$$

From (7), (8), and (9) we obtain (6).

Using (6), we obtain

$$\begin{aligned} \int_{|X| < N} |\varphi(X)| dX &= \sum_{k=0}^{\infty} \int_{E_{k,N}} |\varphi(X)| dx \leq \sum_{k=0}^{\infty} \sqrt{k+1} N \text{mes}_n E_{k,N} \leq \\ &\leq N(2N)^n + \sum_{k=1}^{\infty} \sqrt{\frac{k+1}{k}} 2(\sqrt{2\pi})^{n-1} \exp\{-(k-1)N^2/2\} = O(N^{n+1}). \end{aligned}$$

Hence (5) follows easily.

We shall first prove Lemma 2 in the one-dimensional case. Let $\varphi(z)$, $z \in C^1$, be an entire function satisfying the hypotheses of the lemma, i.e.

$$\liminf_{r \rightarrow \infty} r^{-1} \ln M(r, \varphi) = 0, \quad \int_{|x| > 1} |\varphi(x)| |x|^{-3} dx < \infty. \quad (9)$$

It follows from the second condition that if $\varphi(z)$ is a polynomial, then $\varphi(z)$ is a linear function. We shall suppose that the function $\varphi(z)$ is transcendental.

First we show that the set of zeros of the function $\varphi(z)$ is infinite. Indeed, otherwise one could specify a polynomial $p(z)$ such that the function $\varphi_1(z) = \varphi(z)/p(z)$ has no zeros at all. The function $\varphi_1(z)$, obviously, satisfies the condition

$$\liminf_{r \rightarrow \infty} r^{-1} \ln M(r, \varphi_1) = 0.$$

It is known ((²), p. 48, (³), p. 51) that every entire function $\varphi_1(z)$ satisfying such a condition and having no zeros is constant. We arrive at the conclusion that the function $\varphi(z)$ is a polynomial, which contradicts the assumption of its transcendentality.

Now let a_1, a_2, a_3 be arbitrary three zeros of the function $\varphi(z)$. Consider the entire function

$$\psi(z) = \int_0^z \psi_1(\xi) d\xi, \quad \psi_1(\xi) = \varphi(\xi)/[(\xi - a_1)(\xi - a_2)(\xi - a_3)].$$

It follows from (9) that the function $\psi(z)$ satisfies the conditions

$$\liminf_{r \rightarrow \infty} r^{-1} \ln M(r, \psi) = 0, \quad \sup_{-\infty < x < \infty} |\psi(x)| < \infty. \quad (10)$$

By the Phragmén-Lindelöf theorem ([4], p. 211), every entire function satisfying the conditions (10) is bounded in each of the half-planes $\operatorname{Im} z \geq 0$ and $\operatorname{Im} z \leq 0$ and, consequently, by Liouville's theorem is constant. Since $\varphi(z) = \psi(z)(z - a_1)(z - a_2)(z - a_3)$, we arrive at the conclusion that $\varphi(z) \equiv 0$, which contradicts the transcendence of the function $\varphi(z)$.

We now prove Lemma 2 in the case of dimension $n \geq 2$. Denote by S^n the unit sphere in R^n and put

$$\varphi_{\Theta}(z) = \varphi(z\Theta), \quad z \in C^1, \quad \Theta \in S^n.$$

Denoting by $d\Theta$ the element of surface area of the sphere S^n and taking into account that $\varphi_{-\Theta}(z) = \varphi_{\Theta}(-z)$, we have

$$\begin{aligned} \int_{|X|>1} |\varphi(X)| \cdot |X|^{-n-2} dX &= \int_{S^n} d\Theta \int_1^{\infty} |\varphi_{\Theta}(x)| x^{-3} dx = \\ &= \frac{1}{2} \int_{S^n} d\Theta \int_{|x|>1} |\varphi_{\Theta}(x)| \cdot |x|^{-3} dx. \end{aligned}$$

Since, by the condition of the lemma, the integral (5) converges, it follows that for all $\Theta \in S^n$, except, at most, for a set of measure 0 on S^n , the integral

$$\int_{|x|>1} |\varphi_{\Theta}(x)| \cdot |x|^{-3} dx$$

will converge. Observing that $\liminf_{r \rightarrow \infty} r^{-1} \ln M(r, \varphi_{\Theta}) = 0$, and applying the one-dimensional case of Lemma 2, we conclude that the function $\varphi_{\Theta}(z)$ is linear for all $\Theta \in S^n$, except, at most, for a set of measure 0.

Let

$$\varphi(Z) = \sum_{k=0}^{\infty} p_k(Z)$$

be the expansion of the function $\varphi(Z)$ into a series of homogeneous polynomials. Then

$$\varphi(z\Theta) = \sum_{k=0}^{\infty} p_k(\Theta) z^k.$$

Consequently, $p_k(\Theta) = 0$ almost everywhere on S^n for $k = 2, 3, \dots$. Therefore $p_k(Z) \equiv 0$, $k = 2, 3, \dots$, and the function $\varphi(Z)$ is linear. Lemmas 1 and 2, and together with them Theorem 1, are proved.

The class Ξ admits a generalization. Let X be the random vector discussed at the beginning of the article. Denote by Ξ_m , $1 \leq m \leq n$, the class of vector-functions $W = \Phi(Z) = (\varphi_1(Z), \dots, \varphi_m(Z))$, where $Z \in C^n$, and the $\varphi_j(Z)$ are entire functions, real for $Z = X \in R^n$, satisfying the following condition: the components of the random vector $Y = \Phi(X)$ are independent and distributed according to the law $N(0, 1)$. Obviously, $\Xi_1 = \Xi$, and the class Ξ_m can be characterized by the condition

$$\left(\frac{1}{\sqrt{2\pi}}\right)^m \int_B e^{-|Y|^2/2} dY = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\Phi^{-1}(B)} e^{-|X|^2/2} dX, \quad (11)$$

where B is any Borel set in R^m , and $\Phi^{-1}(B)$ is its full preimage in R^n . Putting in (11) $B = \{Y : Y = (y_1, \dots, y_m) \in R^m, y_j \in A\}$, where A is any Borel set in R^1 , and observing that then $\Phi^{-1}(B) = \varphi_j^{-1}(A)$, we are convinced of the validity of the following assertion (Yu. V. Linnik): if $\Phi(Z) = (\varphi_1(Z), \dots, \varphi_m(Z)) \in \Xi_m$, then $\varphi_j(Z) \in \Xi$, $j = 1, \dots, m$. With the aid of this assertion, from Theorem 1 we obtain a more general fact.

Theorem 2. Let $\Phi(Z) = (\varphi_1(Z), \dots, \varphi_m(Z)) \in \Xi_m$. Then about each of the functions $\varphi_j(Z)$, $1 \leq j \leq m$, one may assert the following: either

$\varphi_j(Z)$ is a linear function of the form (2), or else $\varphi_j(Z)$ satisfies the condition

$$\liminf_{r \rightarrow \infty} r^{-1} \ln M(r, \varphi_j) > 0.$$

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References

1. Yu. V. Linnik, V. L. Eidlin, *Theory of Probability and Its Applications*, **13**, no. 4, 751 (1968).
2. U. Hayman, *Meromorphic Functions*, Moscow, 1966.
3. A. A. Gol'dberg, I. V. Ostrovskii, *Distribution of Values of Meromorphic Functions*, Moscow, 1970.
4. A. I. Markushevich, *Theory of Analytic Functions*, **2**, Moscow, 1968.

Note: Figure translations are in progress. See original paper for figures.

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