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V. Ya. Gutlyanskii

1970

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Abstract

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UDC 517:54

MATHEMATICS

V. Ya. Gutlyanskii

PARAMETRIC REPRESENTATION OF UNIVALENT FUNCTIONS

(Presented by Academician M. A. Lavrent'ev on 24 III 1970)

Let S be the class of all holomorphic functions $w = f(z)$ univalent in the disk $E = \{z : |z| < 1\}$, normalized by the conditions $f(0) = 0$, $f'(0) = 1$. In the present paper we give a solution of the problem of the parametric representation of the class S , establishing necessary and sufficient conditions for a function $w = f(z)$ to belong to the class S . We then indicate one application of the result obtained to the solution of extremal problems in the theory of univalent functions and note the connection between extremal problems in the class S and the class P of all functions $w = h(z)$ holomorphic in the disk E , $h(0) = 1$, with positive real part.

1. Let $\mathfrak{M}$ be the class of all nondecreasing functions $\mu(x, y)$ of two variables in the domain $x \geq 0$, $-\pi \leq y \leq \pi$, normalized by the conditions $\mu(x, -\pi) = \mu(0, y) = 0$, $\mu(x, \pi) = x$.

It follows immediately from the definition of the class $\mathfrak{M}$ that for each fixed y , $-\pi \leq y \leq \pi$, the functions $\mu(x, y)$ are absolutely continuous with respect to x , and consequently, for almost all x , $x > 0$, there exists the derivative $\mu'_x(x, y)$, which is a measurable function of the variable x for each fixed value of y , and a nondecreasing function of the variable y , $-\pi \leq y \leq \pi$, for fixed x , $x > 0$, normalized by the condition $\mu'_x(x, -\pi) = 0$, $\mu'_x(x, \pi) = 1$.

We shall say that a sequence $\mu_n(x, y)$ ($n = 1, 2, \dots$) of functions of the class $\mathfrak{M}$ converges to a function $\mu(x, y) \in \mathfrak{M}$ if at all points of continuity of the function $\mu(x, y)$ one has $\lim_{n \rightarrow \infty} \mu_n = \mu$.

The class $\mathfrak{M}$ is compact in itself with respect to the convergence of sequences of functions from $\mathfrak{M}$ defined above.

Denote by Φ the set of all continuous functions $f(z, x, y)$ in the domain $E \times [0, \infty) \times [-\pi, \pi]$, analytic with respect to z in the disk E and satisfying the condition $|f(z, x, y)| \leq e^{-x} K(r)$, where $K(r)$ is a constant depending only on $r = |z| < 1$.

Let $f(z, x, y)$ be an arbitrary function of the class Φ , and let μ_n be an arbitrary

sequence of functions of the class $\mathfrak{M}$ converging to a function $\mu(x, y) \in \mathfrak{M}$. Then:

- 1) there exists, uniformly with respect to x , $0 \leq x \leq A$, and $z \in E_r = \{z : |z| \leq r < 1\}$, the limit

$$\lim_{n \rightarrow \infty} \int_0^x \int_{-\pi}^{\pi} f(z, x, y) d\mu_n(x, y) = \int_0^x \int_{-\pi}^{\pi} f(z, x, y) d\mu(x, y);$$

- 2) the Stieltjes integrals

$$\int_0^{\infty} \int_{-\pi}^{\pi} f(z, x, y) d\mu(x, y), \quad \mu \in \mathfrak{M},$$

converge uniformly inside E , uniformly with respect to the class $\mathfrak{M}$.

From this there follows directly the existence, uniformly inside E , of the limit

$$\lim_{n \rightarrow \infty} \int_0^{\infty} \int_{-\pi}^{\pi} f(z, x, y) d\mu_n(x, y) = \int_0^{\infty} \int_{-\pi}^{\pi} f(z, x, y) d\mu(x, y).$$

2. Consider the differential equation

$$\frac{dw}{dx} = -w \int_{-\pi}^{\pi} g(w, y) d\mu'_x(x, y), \quad (1)$$

where $g(w, y) = (1 + e^{iy}w)/(1 - e^{iy}w)$, with the initial condition

$$w(x)|_{x=0} = z, \quad z \in E.$$

Here the function $\mu(x, y) \in \mathfrak{M}$, and the integral in (1) is understood in the Stieltjes sense.

We shall denote by $f(z, x; \mu)$ the solution of the differential equation (1) satisfying the initial condition.

Theorem 1. *In order that the function $w = f(z)$ belong to the class S , it is necessary and sufficient that it can be represented in the form*

$$f(z) = \lim_{x \rightarrow \infty} e^x f(z, x; \mu), \quad \mu \in \mathfrak{M}. \quad (2)$$

We outline the proof of Theorem 1. Let $\mu(x, y)$ be an arbitrary function of the class $\mathfrak{M}$. Replace equation (1) with the initial condition by the integral equation

$$w = z \exp \left\{ - \int_0^x \int_{-\pi}^{\pi} g(w, y) d\mu(x, y) \right\}, \quad (3)$$

which is obtained from (1) by division by w and integration with respect to x from 0 to x . Solving (3) by the method of successive approximations (cf., for example, ⁽¹⁾, pp. 96–97), we find that the solution $w = f(z, x; \mu)$ of equation (3) is regular in the disk E and continuous for $0 < x < \infty$, and, moreover,

$$f(0, x; \mu) = 0, \quad f'_z(0, x; \mu) = e^{-x}.$$

By virtue of an easily proved uniqueness theorem for the solution of equation (1), it follows that the function $f(z, x; \mu)$ is univalent in E for each fixed value of x in $[0, \infty)$. It remains to establish the existence, uniformly in z inside E , of the limit (2). For this purpose we substitute $f(z, x; \mu)$ into equation (1) and rewrite it in the form

$$[e^x f(z, x; \mu)]'_x = e^x f(z, x; \mu) [1 - g(f(z, x; \mu), y)], \quad (4)$$

noting at the same time that the function standing on the right-hand side of equation (4) belongs to the class Φ^* .

Integrating (4) with respect to x from 0 to x and passing to the limit as x tends to infinity, we arrive at the conclusion that the function $f(z)$ obtained from formula (2) belongs to the class S .

Now let $f(z)$ be an arbitrary function of the class S . We shall show that it can be obtained by formula (2) with a suitably chosen function $\mu(x, y)$ from the class $\mathfrak{M}$. To this end denote by $\mathfrak{M}'$ the subclass of the class $\mathfrak{M}$ consisting of functions $\mu(x, y)$ such that

$$\int_{-\pi}^{\pi} g(w, y) d\mu'_x(x, y) = g(w, y(x)).$$

By Loewner's theorem ⁽²⁾ (see also ⁽¹⁾, p. 95), the totality of functions $f(z)$ obtained by formula (2), when $\mu(x, y)$ runs through the class $\mathfrak{M}'$, forms a subclass S' of the class S , everywhere dense in S with respect to uniform convergence inside the disk E .

* This follows from the estimates $|f(z, x; \mu)| \leq |z|$, $|f(z, x; \mu)| \leq \frac{e^{-x}|z|}{(1 - |z|)^2}$.

Choose a sequence $f_n(z)$ of functions of the class S' , converging uniformly inside E to the function $f(z)$. To the sequence $f_n(z)$ there corresponds a sequence $\mu_n(x, y)$ of functions of the class $\mathfrak{M}$ such that

$$f_n(z) = \lim_{x \rightarrow \infty} e^x f(z, x; \mu_n).$$

From $\mu_n(x, y)$ one can choose a subsequence converging, in the sense indicated earlier, to some function $\mu^*(x, y)$ of the class $\mathfrak{M}$. Now, using the propositions of Section 1, it is not difficult to show that the function $f(z)$ itself can be obtained by formula (2) for $\mu = \mu^*$.

From the Riesz-Herglotz theorem ⁽³⁾ it follows that the function

$$h(w, x) = \int_{-\pi}^{\pi} g(w, y) d\mu'_x(x, y), \quad \mu \in \mathfrak{M}, \quad (5)$$

for each fixed x , $0 < x < \infty$, is regular in w in the disk $|w| < 1$ and has there a positive real part. Consequently, from the known differential equation of K. Löwner-P. P. Kufarev ⁽⁴⁾ and relation (2), all functions of the class S can be obtained.

3. From the identity

$$dw/dx = -wh(w, x), \quad w = f(z, x; \mu), \quad (6)$$

where $h(w, x)$ is computed by formula (5), taking (2) into account, there immediately follow the relations in the class S that are needed for the subsequent arguments:

$$f(z) = z \exp \left\{ \int_0^{|z|} \frac{1 - F(w, \rho)}{\operatorname{Re} F(w, \rho)} \frac{d\rho}{\rho} \right\}, \quad (7)$$

$$f'(z) = \exp \left\{ \int_0^{|z|} \frac{1 - F(w, \rho) - wF'_w(w, \rho)}{\operatorname{Re} F(w, \rho)} \frac{d\rho}{\rho} \right\}. \quad (8)$$

Here $F(w, \rho) = h(f(z, x(\rho); \mu), x(\rho))$, $\rho = |f(z, x; \mu)|^*$.

Theorem 2. Let z_0 be a fixed point of the disk E , and let $\alpha, \beta, \gamma, \delta$ be arbitrary real numbers. Then, for the functional

$$I(f) = \alpha \ln \left| \frac{f(z_0)}{z_0} \right| + \beta \arg \frac{f(z_0)}{z_0} + \gamma \ln |f'(z_0)| + \delta \arg f'(z_0), \quad (9)$$

defined on the class S , the following sharp estimates hold:

$$\int_0^{|z_0|} \varphi(\xi^-, \eta^-) \frac{d\rho}{\rho} \leq I(f) \leq \int_0^{|z_0|} \varphi(\xi^+, \eta^+) \frac{d\rho}{\rho}, \quad (10)$$

where $(\xi^\pm, \eta^\pm)$ are the points of the circle

$$\xi^2 - 2a(\rho)\xi + \eta^2 + 1 = 0, \quad a = (1 + \rho^2)(1 - \rho^2)^{-1},$$

at which the function

$$\varphi(\xi, \eta) = a - \alpha - \gamma + (\alpha + \gamma)/\xi - \gamma\xi - \eta(\delta + (\beta + \delta)/\xi) \quad (11)$$

attains its maximum (minimum) value.

Equality in (10) is realized, for example, for functions $f(z)$ of the class S having the form

$$f(z) = \lim_{x \rightarrow \infty} e^x f(z, x),$$

where $w = f(z, x)$ is the solution of the equation

$$w'_x = -wg(w, y^\pm(x)), \quad w(0) = z,$$

where

$$y^\pm(x) = \arcsin \eta^\pm [\xi^\pm (a^2 - 1)^{1/2}]^{-1} + \int_0^x \eta^\pm dx - \arg z_0,$$

and $\rho = \rho(x)$ is determined from the relation

$$(\ln \rho)'_x = -\xi^\pm, \quad \rho(0) = |z_0|.$$

* From (6) it follows that $\rho(x)$ is a monotonically decreasing function, since $(\ln \rho)'_x = -\operatorname{Re} h < 0$.

From formulas (7), (8) it follows that the problem of estimating the functional $I(f)$ of the form (9) on the class S is equivalent to finding the extremum of the real functional $J(h) = \Psi(h(z), zh'(z))/\operatorname{Re} h(z)$, $z = \rho e^{i\varphi} \in E$ and fixed, where $\Psi(\omega, w) = (\alpha + \gamma)(1 - \operatorname{Re} \omega) - (\beta + \delta) \operatorname{Im} \omega - \gamma \operatorname{Re} w - \delta \operatorname{Im} w$, on the class P (see, on this question, works ^{5,6}) and to the subsequent integration of the result with respect to ρ from 0 to $|z_0|$. A similar connection between the extremal problems of the classes S and P also holds for other problems in the theory of functions of a complex variable.

Donetsk Computing Center
Academy of Sciences of the Ukrainian SSR

Received
18 III 1970

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