

ON INTEGRAL OPERATORS

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Abstract

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MATHEMATICS

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ON INTEGRAL OPERATORS

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1°. Let E be a real B -space, and let $S(a, b)$ be the space of measurable functions. In 1938 L. V. Kantorovich and B. Z. Vulikh ⁽¹⁾ introduced and studied a class of operators acting from E into $S(a, b)$ and having the form

$$(T(x))(s) = (x, \varphi(s)), \quad x \in E, \quad (1)$$

where $\varphi(s)$ is a function* with values in the conjugate space E^* .

For separable or reflexive B -spaces E , L. V. Kantorovich and B. Z. Vulikh showed that an operator $T : E \rightarrow S(a, b)$ has the form (1) if and only if T is a bo -linear operator ⁽²⁾.

Operators of the form (1) are a generalization of integral operators: if E and E^* are spaces of measurable functions and $(x, x^*) = \int x(t)x^*(t) dt$, then (1) has the form

$$(Tx)(s) = \int K(s, t)x(t) dt, \quad x \in E, \quad (2)$$

where the kernel $K(s, t) = \varphi(s)$ is measurable in t for almost all $s \in (a, b)$ and satisfies the condition $K(s, \cdot) \in E^*$, $s \in (a, b)$.

It is known ⁽¹⁻³⁾ that, under additional restrictions on E and on the operator T , the kernel in (2) is jointly measurable in the variables; in ^(1, 2) it was shown that if the operator T is bo -linear from $L_p(a, b)$ into $L_q(a, b)$, then the kernel in (2) is jointly measurable in the variables and, moreover, satisfies the condition $\| \| K(s, t) \|_{p'} \|_q < \infty$, $1/p + 1/p' = 1$.

In ⁽³⁾ this result was generalized to the case of bo -linear operators from $X(0, 1)$ into $Y(0, 1)$, where $X(0, 1), Y(0, 1)$ are spaces of measurable functions belonging respectively to the classes P and $P \cup Q$, introduced by D. A. Vladimirov ⁽³⁾.

It should be noted that the proofs of the propositions cited from ⁽¹⁻³⁾ rely essentially on the separability of the space from which the operator acts.

The difficulties that arise in representing the function $\varphi(s)$ in the form of a jointly measurable kernel, and the restrictions on spaces and operators associated with them, are due to the fact that the function $\varphi(s)$ in (1) is weakly measurable. These difficulties largely disappear if the function $\varphi(s)$ is strongly measurable^(4, 5).

In this connection there arises the problem of conditions for representability of an operator in the form (1) with a strongly measurable function $\varphi(s)$. Below a solution of this problem is given for operators acting from an arbitrary normed space (real or complex) into the space $S(X, \mu)$, where (X, μ) is an arbitrary space with a σ -finite measure. Further, a broad class of spaces $F(Y, \nu)$ of ν -measurable functions is indicated ((Y, ν) is an arbitrary space with a σ -finite measure) possessing the property that strongly measurable functions $\varphi(s)$ with values in these spaces generate, by the equality $\varphi(s) = K(s, \cdot)$, kernels measurable jointly in the variables, and propositions are given on

* From (1) it follows that the function $\varphi(s)$ is weakly measurable.

integral representability of linear operators acting from normed spaces of measurable functions into spaces of measurable functions.

2°. Let E be a normed space (real or complex), L a linear manifold in E , and (X, Σ, μ) a space with a σ -finite measure. A linear operator $T : L \rightarrow S(X, \mu)$ will be called a C -operator if

$$(Tx)(s) = (x, \varphi(s)), \quad x \in L,$$

where $\varphi(s) : X \rightarrow E^*$ is a strongly μ -measurable function, which we shall call the kernel of the operator T . In what follows we shall assume that the closure of L has a topological complement in E ⁽⁶⁾.

We shall call a set $H \subset L^\infty(X, \mu)$ σ -weakly compact if there exists an at most countable set of disjoint sets X_n , $n \in I$, such that $\mu(X_n) < \infty$, $\mu(X \setminus \bigcup_{n \in I} X_n) = 0$, and the sets $P_n(H)$, $n \in I$, are weakly compact in $L^\infty(X, \mu)$, where

$$P_{nf} = \chi_{X_n} f,$$

χ_{X_n} is the characteristic function of the set X_n .

A linear bounded operator $Q : L \rightarrow L^\infty(X, \mu)$ will be called σ -weakly completely continuous if, for every bounded set $G \subset L$, the set $Q(G)$ is σ -weakly compact in $L^\infty(X, \mu)$.

Theorem 1. *A linear operator $T : L \rightarrow S(X, \mu)$ is a C -operator if and only if there exists a function $\Lambda \in S(X, \mu)$, $\Lambda > 0$, such that the operator*

$$(\tau x)(s) = (Tx)(s)/\Lambda(s), \quad x \in L,$$

is a σ -weakly completely continuous operator from L into $L^\infty(X, \mu)$.

Remark. If L is everywhere dense in E , then the kernel of the operator T is determined uniquely up to μ -equivalence.

Corollary. Let $E = H$, where H is a Hilbert space. An operator $T : H \rightarrow L_2(X, \mu)$ is a Hilbert-Schmidt operator ⁽⁷⁾ if and only if the operator T has a majorant ⁽⁸⁾ belonging to $L_2(X, \mu)$.

In the case when H and $L_2(X, \mu)$ are separable, the corollary coincides with Proposition 2.10 ⁽⁹⁾.

Theorem 2. Let E be reflexive, and let $T : L \rightarrow S(X, \mu)$ be a linear operator. The following assertions are equivalent: 1) T has an abstract norm ⁽²⁾; 2) T has a majorant ⁽⁸⁾; 3) T is a C -operator; 4) T has the form

$$(Tx)(s) = (x, \varphi(s)),$$

where φ is weakly μ -measurable; T is a b_0 -linear operator, i.e., it maps null sequences from L into sequences converging to zero μ -almost everywhere.

3°. Let (Y, Ξ, ν) be a space with a σ -finite measure, and let $F(Y, \nu)$ be a normed space of ν -measurable functions*. We shall say that $F(Y, \nu)$ is σ -embedded in $L_1(Y, \nu)$ if there exists an at most countable set of disjoint sets Y_n , $n \in I$, such that $\mu(Y_n) < \infty$,

$$\mu\left(Y \setminus \bigcup_{n \in I} Y_n\right) = 0$$

and

$$\|P_{n_f}\|_{L_1(Y, \nu)} \leq c_n \|f\|_{F(Y, \nu)},$$

where $P_{n_f} = \chi_{Y_n} f$ and c_n does not depend on f .

Lemma. Let $\varphi(s) : X \rightarrow F(Y, \nu)$ be a strongly μ -measurable function and let $F(Y, \nu)$ be σ -embedded in $L_1(Y, \nu)$. Then there exists a $(\mu \times \nu)$ -measurable function $K(s, t)$ such that $\varphi(s) = K(s, \cdot)$.

Theorem 3. Let $G(Y, \nu)$, $[G(Y, \nu)]^*$ be spaces of ν -measurable functions and

$$(x, x^*) = \int_Y x(t) \overline{x^*(t)} d\nu(t), \quad x \in G(Y, \nu), \quad x^* \in [G(Y, \nu)]^*,$$

$[G(Y, \nu)]^*$ is σ -embedded in $L_1(Y, \nu)$, and L is a linear manifold in $G(Y, \nu)$. If the operator $T : L \rightarrow S(X, \mu)$ is a C -operator, then the operator T is an integral operator with a $(\mu \times \nu)$ -measurable kernel $K(s, t)$, satisfying the condition

$$\|K(s, \cdot)\|_{[G(Y, \nu)]^*} \in S(X, \mu).$$

* By a space $Z(Y, \nu)$ of measurable functions, here and below in the article we mean a space of classes f of ν -equivalent functions satisfying the conditions: a) if $f \in Z(Y, \nu)$, then also $|f| \in Z(Y, \nu)$, and $\|f\|_Z = \|f\|_Z$; b) if $f \in Z$ and $|g| \leq |f|$, $g \in S(Y, \nu)$, then $g \in Z$ and $\|g\|_Z \leq \|f\|_Z$.

4°. **Theorem 4.** Let 1) $H(Y, \nu)$, $G(Y, \nu)$, $F(X, \mu)$ be normed spaces of measurable functions; 2) $L = H(Y, \nu) \cap G(Y, \nu)$ be everywhere dense in $H(Y, \nu)$ and $G(Y, \nu)$; 3) $G(Y, \nu)$ satisfy the conditions of the preceding Theorem 3 and

be reflexive; 4) $H(Y, \nu)$ be a Banach space; 5) $T : H(Y, \nu) \rightarrow F(X, \mu)$ be a regular operator. In order that the operator T be an integral operator with a $(\mu \times \nu)$ -measurable kernel satisfying the condition $\|K(s, \cdot)\|_{[G(Y, \nu)]^*} \in S(X, \mu)$, it is necessary and sufficient that there exist a function $\Lambda \in S(X, \mu)$ such that, for all $f \in L$, $|(Tf)(s)| \leq \Lambda(s)\|f\|_{G(Y, \nu)}$ for μ -almost all $s \in X$.

5°. The results obtained in ^(8,9) for operators in separable spaces can be extended to the nonseparable case. Thus, assertions 1), 3) of Theorems 1–5 ⁽⁸⁾ remain valid also in the case when μ, μ_0 are not separable * σ -finite measures. Assertions 2) of Theorems 1–5 ⁽⁸⁾ are formulated taking into account the separability of the range of operators of type (SC) . We give, for example, analogues of assertions 2) of Theorems 1, 2 ⁽⁸⁾.

Theorem 5. Let the measure μ be σ -finite and not purely atomic. An operator T is an operator of type (SC) if and only if the adjoint operator T^* is densely defined, the residual spectrum of the operator T^* contains 0, and the range of the closure of the operator T is separable.

Theorem 6. Let the measure μ be σ -finite and not purely atomic. An operator T is an operator of type (C) if and only if the adjoint operator T^* is densely defined and there exists a symmetric operator A such that $A \subseteq T^*$, the residual spectrum of the operator A contains 0, and the range of the adjoint operator A^* is separable.

6°. **Theorem 7.** An operator $T : L_2(Y, \nu) \rightarrow L_2(X, \mu)$ is an integral operator with a $(\mu \times \nu)$ -measurable Hilbert–Schmidt kernel if and only if there exists a majorant ⁽⁸⁾ of the operator T belonging to $L_2(X, \mu)$.

In the case $X = Y = (a, b)$, $\mu = \nu$ is Lebesgue measure, the theorem follows from a theorem of L. V. Kantorovich and B. Z. Vulikh (⁽²⁾, p. 332; see also ⁽³⁾, p. 773).

Corollary 1. Let $A : L_2(Z, \xi) \rightarrow L_2(Y, \nu)$ and $B : L_2(X, \mu) \rightarrow L_2(W, \eta)$ be linear bounded operators, and let $T : L_2(Y, \nu) \rightarrow L_2(X, \mu)$ be an integral operator with a $(\mu \times \nu)$ -measurable Hilbert–Schmidt kernel. Then BTA is an integral operator with an $(\eta \times \xi)$ -measurable Hilbert–Schmidt kernel.

Corollary 2. An operator $T : L_2(Y, \nu) \rightarrow L_2(X, \mu)$ is a Hilbert–Schmidt operator ⁽⁷⁾ if and only if the operator T is an integral operator with a $(\mu \times \nu)$ -measurable Hilbert–Schmidt kernel.

In the separable case the assertion of the corollary is well known (⁽⁴⁾, p. 102).

7°. The results of the present article can be extended to the case where X is a separable locally compact space and μ is a Radon measure.

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* That is, $L_2(X, \mu)$, $L_2(X_0, \mu_0)$ are not separable.

Note: Figure translations are in progress. See original paper for figures.

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