

LIMIT THEOREMS FOR FUNCTIONALS OF THE PROCESS OF STEP SUMS OF RANDOM VARIABLES DEFINED ON A SEMI-MARKOV PROCESS WITH A FINITE SET OF STATES

D. S. Sil'vestrov

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Abstract

Full Text

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MATHEMATICS

D. S. SILVESTROV

LIMIT THEOREMS FOR FUNCTIONALS OF THE PROCESS OF STEP SUMS OF RANDOM VARIABLES DEFINED ON A SEMI-MARKOV PROCESS WITH A FINITE SET OF STATES

(Presented by Academician A. N. Kolmogorov on 11 V 1970)

Let T_j , $j = 1, 2$, be independent collections of random variables, defined as follows: $T_1 = \{\eta_n, n = 0, 1, 2, \dots\}$ is a homogeneous Markov chain with finite set of states $H = \{1, 2, \dots, m\}$ and transition-probability matrix $\|p_{ij}\|_{i,j=1}$; $T_2 = \{(\tau(n, i), \gamma(n, i)), n \geq 0, i \in H\}$ is a collection of independent random vectors, taking values in $[0, \infty) \times (-\infty, \infty)$, whose distributions do not depend on n .

The random process

$$\eta(t) = \eta_{\nu(t)}, \quad t \geq 0,$$

where

$$\nu(t) = \max \left(n : \sum_{k=1}^n \tau(k-1, \eta_{k-1}) \leq t \right),$$

is called a semi-Markov process ⁽¹⁾.

We require that T_j , $j = 1, 2$, satisfy the following regularity condition:

(A_1) : 1) T_1 is ergodic (we denote its stationary distribution by q_j , $j = 1, 2, \dots, m$);

$$2) \sum_{i=1}^m P\{\tau(0, i) > 0\} > 0.$$

For each $t > 0$, let us introduce the random process

$$\xi_t(s) = \sum_{k=1}^{\nu(st)} \gamma(k-1, \eta_{k-1}), \quad s \in [0, 1],$$

which it is natural to call the process of step sums of random variables defined on the semi-Markov process $\eta(t)$, $t \geq 0$.

Let $D_{[0,1]}$ be the space of functions on $[0, 1]$ without discontinuities of the second kind, right-continuous with the uniform metric

$$\rho(x(s), y(s)) = \sup_{s \in [0,1]} |x(s) - y(s)|,$$

$$\mu(C) = P\{w(s) \in C\}, \quad C \in \mathfrak{B};$$

here $\mathfrak{B}$ is the σ -algebra of Borel sets in $D_{[0,1]}$, and $w(s)$, $s \in [0, 1]$, is a Wiener process continuous with probability 1.

It is obvious from the construction that for all $t > 0$, with probability 1 the trajectories of the random process $\xi_t(s)$, $s \in [0, 1]$, belong to $D_{[0,1]}$.

Definition. We shall say that a measurable functional $f(\cdot)$, defined on $D_{[0,1]}$, is continuous in the uniform topology if there exists a set $C \in \mathfrak{B}$ such that $\mu(C) = 1$ and for all $x_n(s) \in D_{[0,1]}$,

$n \geq 0$, if

$$x_0(s) \in C, \quad \rho(x_n(s), x_0(s)) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then

$$\lim_{n \rightarrow \infty} f(x_n(s)) = f(x_0(s)).$$

A number of examples of μ -continuous functionals in the uniform topology are given in ⁽²⁾.

Theorem 1. *If condition (A_1) and $(A_2)'$ are satisfied: $M|\gamma(0, i)|^2 < \infty$, $M\tau(0, i)^2 < \infty$, $i \in H$, then all finite-dimensional distributions of the random process*

$$w_t(s) = t^{-1/2} \left(\xi_t(s) - \frac{b}{a} st \right), \quad s \in [0, 1],$$

as $t \rightarrow \infty$, converge weakly (at continuity points) to the corresponding finite-dimensional distributions of the random process

$$w_0(s) = \sigma w(s), \quad s \in [0, 1];$$

here

$$a = \sum_{i=1}^m q_i M \tau(0, i), \quad b = \sum_{i=1}^m q_i M \gamma(0, i),$$

$$\sigma^2 = \lim_{n \rightarrow \infty} (an)^{-1} D \sum_{k=1}^n \left(\gamma(k-1, \eta_{k-1}) - \frac{b}{a} \tau(k-1, \eta_{k-1}) \right).$$

Remark 1. In ⁽³⁾ a simple method is given for finding the constant through $\|p_{ij}\|_{i,j=1}^m$ and $M\gamma(0, i)^{k'} \tau(0, i)^{k''}$, $i \in H$, $k', k'' \geq 0$, $k' + k'' \leq 2$, reducing to the solution of a finite system of linear equations.

Theorem 2. If the conditions (A_j) , $j = 1, 2$, are satisfied, then for all functionals $f(\cdot)$ on $D_{[0,1]}$ that are μ -continuous in the uniform topology,

$$P\{f(w_t(s)) < u\} \rightarrow P\{f(w_0(s)) < u\} \quad \text{as } t \rightarrow \infty$$

for all continuity points of the distribution function standing on the right.

Remark 2. For the case when condition (B) is satisfied:

- 1) $\tau(0, i) = 1$, $i \in H$ with probability 1;
- 2) $m = 1$ (control by the Markov chain is absent).

The corresponding results are contained, for example, in ⁽²⁾.

The proof of Theorem 1 is carried out analogously to how this is done for one-dimensional distributions in ⁽⁴⁾.

The proof of Theorem 2 contains two stages.

It is not difficult to prove that, when the conditions (A_j) , $j = 1, 2$, are satisfied,

$$\sup_{s \in [0,1]} \left| \frac{\nu(st)}{t} - a^{-1}s \right| \xrightarrow{P} 0 \quad \text{as } t \rightarrow \infty.$$

Next, using a simple assertion,

Lemma 1. If, for a sequence of random processes $\xi_n(s)$, $s \geq 0$, $n = 0, 1, \dots$, whose trajectories with probability 1 belong to the space $D_{[0,\infty)}$ of functions on $[0, \infty)$ without discontinuities of the second kind and continuous from the right, and a sequence of random processes $\nu_n(s)$, $s \geq 0$, $n = 0, 1, \dots$, taking nonnegative values with probability 1 and whose trajectories with probability 1 belong to $D_{[0,\infty)}$, the relations

$$\text{a) } \rho_n(t) = \sup_{s \in [0, t]} |\xi_n(s) - \xi_0(s)| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty,$$

where $\xi_0(s)$, $s \geq 0$, is a random process continuous with probability 1, $t \geq 0$;

b)

$$\hat{\rho}_n(T) = \sup_{s \in [0, T]} |v_n(s) - v_0(s)| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty, \quad T \geq 0;$$

c)

$$P \left\{ \sup_{s \in [0, T]} v_0(s) \geq t \right\} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

then

$$\sup_{s \in [0, T]} |\xi_n(v_n(s)) - \xi_0(v_0(s))| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty;$$

and by the identity

$$\{v(t) \geq x\} = \left\{ \sum_{k=1}^{[x]} \tau(k-1, \eta_{k-1}) \leq t \right\},$$

the proof can be reduced to the case where all $\tau(0, i) = 1$, $i \in H$, with probability 1.

The remainder of the proof is carried out analogously to that given in ² for the case in which condition (B) holds.

Kyiv State University
named after T. G. Shevchenko

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Note: Figure translations are in progress. See original paper for figures.

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