

On a Necessary Condition for Tikhonov Regularizability

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Abstract

Full Text

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MATHEMATICS

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ON ONE NECESSARY CONDITION FOR TIKHONOV REGULARIZABILITY

(Presented by Academician A. N. Tikhonov on 17 IV 1970)

Suppose we have a mapping A of a metric space into a metric space U , with $Az_1 \neq Az_2$ if $z_1 \neq z_2$. The problem of approximate solution of the equation

$$Az = u \quad (1)$$

is called Tikhonov-regularizable ⁽¹⁾ if there exists a one-parameter family of mappings $R_\delta : U \rightarrow Z$, $0 < \delta \leq 1$, for which

$$\lim_{\delta \rightarrow 0} \sup_{\{u \in U: \rho(u, Az) \leq \delta\}} \rho(z, R_\delta u) = 0, \quad \forall z \in Z. \quad (2)$$

Theorem. If problem (1) is Tikhonov-regularizable and N is an everywhere dense set in U , then

$$\bigcup_{n=1}^{\infty} \{R_{1/n} N\}$$

is an everywhere dense set in Z .

Proof. For an arbitrary point $z \in Z$ and any natural n , take $u_n \in N$ such that $\rho(Az, u_n) \leq 1/n$. By the regularizability condition (2),

$$\lim_{n \rightarrow \infty} \rho(z, R_{1/n} u_n) = 0,$$

i.e. the sequence $\{R_{1/n} u_n\}$ converges to the element z of the space Z . Since $u_n \in N$, the theorem is proved.

Corollary. If problem (1) is Tikhonov-regularizable and in the metric space U there exists an infinite everywhere dense set of cardinality τ , then in the metric space Z there also exists an everywhere dense set of cardinality τ . In particular, separability of Z follows from separability of U .

From this, examples of nonregularizable problems easily follow.

1. For the embedding operator of the space of measurable bounded functions on the segment $M_{[0,1]}$ into $L_{2[0,1]}$, problem (1) is not Tikhonov-regularizable.
2. Let Z be a nonseparable metric space of monotonically increasing functions on $[0, 1]$, the distance between which is measured in the metric of the space $M_{[0,1]}$; let U be an arbitrary separable metric space (for example L_2). Then for any mapping $A : Z \rightarrow U$, regularization is impossible, i.e. it is impossible to construct approximate solutions $z_\delta \in Z$ whose deviation from the exact solution $z \in Z$ would tend to zero in the metric of the space $M_{[0,1]}$. Note that A may be a linear continuous or a completely discontinuous operator.
3. The integral equation

$$\int_0^1 k(x, t) dg(t) = u(x),$$

considered from $M_{[0,2\pi]}$ into $L_{2[0,2\pi]}$, is nonregularizable for any kernel $k(x, t)$.

4. The integral equation

$$Az = \int_0^{2\pi} k(x - t)z(t) dt = u(x),$$

considered from $M_{[0,2\pi]}$ into $L_{2[0,2\pi]}$, where $k(t)$ is a 2π -periodic even function satisfying the condition

$$\int_0^{2\pi} k(t) \cos nt dt \neq 0 \quad (n = 0, 1, 2, \dots),$$

is not regularizable. With a suitable choice of $k(t)$ (for example, $k(t) \in L_{2[0,2\pi]}$) the linear operator A will be completely continuous.

The last result shows that A. B. Bakushinskii's question (see ², p. 71) on the possibility of Tikhonov regularization of a linear bounded or completely continuous operator from a Banach space into a Banach space, under the condition of uniqueness of the solution of equation (1), must be answered in the negative. However, the question of regularizability remains open for a separable Banach space Z .

From the corollary to the theorem, together with A. B. Bakushinskii's result on the regularizability of problem (1), if Z is a uniformly convex Banach space, A is a linear bounded operator, $A^{-1}0 = \{0\}$, and U is a Banach space, we obtain an interesting corollary for linear operators in a Banach space: if there exists a linear continuous mapping A of a uniformly convex Banach space Z into a separable Banach space U , $A^{-1}0 = \{0\}$, then Z is separable.

If there exists a completely continuous linear mapping A of a uniformly convex Banach space Z into a normed space, such that $A^{-1}0 = \{0\}$, then Z is separable. This follows from the separability of the range of a completely continuous operator.

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CITED LITERATURE

¹ A. N. Tikhonov, DAN, **153**, No. 1, 49 (1963). ² A. B. Bakushinskii, in the collection *Computational Methods and Programming*, issue 12, Moscow, 1969, p. 56.

Note: Figure translations are in progress. See original paper for figures.

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