

ON THE QUESTION OF AN ASYMPTOTIC FORMULA

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Abstract

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MATHEMATICS

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ON THE QUESTION OF AN ASYMPTOTIC FORMULA

FOR THE NUMBER OF SOLUTIONS OF A CONGRUENCE OF WARING TYPE

(Presented by Academician Yu. V. Linnik on 27 X 1969)

Consider the congruence

$$x_1^n + x_2^n + \dots + x_t^n \equiv d \pmod{p^s}, \quad (1)$$

where n, t, s are natural numbers, p is prime, and d is an arbitrary integer. Let $M_1, \dots, M_t, Q_1, \dots, Q_t$ be integers, $0 \leq M_j < M_j + Q_j \leq p^s, j = 1, \dots, t$. In the works of A. A. Karatsuba ^(1,2) an asymptotic formula was obtained for the number of solutions of congruence (1) in an incomplete system of residues

$$M_j \leq x_j \leq M_j + Q_j - 1, \quad j = 1, \dots, t. \quad (2)$$

Generally speaking, it is impossible to obtain an analogous asymptotic formula for the number of solutions of congruence (1) when $t = n$ (see ^(2,3)). In this connection it is of interest to study such a congruence which, in a certain sense, would differ little from congruence (1), and for the number of whose solutions an asymptotic formula could be obtained for $t = n$. In the present work we consider the congruence obtained from congruence (1) by replacing the constant number d by a variable y , with a small interval of variation q , and some generalizations of it with a possibly smaller number of variables $x_1, \dots, x_t$. A similar approach to the formulation of the problem was used by Yu. V. Linnik ⁽⁴⁾ in considering the binary Goldbach problem. An asymptotic formula for the number of solutions of the congruence obtained in this way already holds for $t = \min(n, s)$ and q increasing arbitrarily slowly with the growth of p , or for fixed $p \geq n + 1$ and q increasing arbitrarily slowly with the growth of s , if $t \geq n + 1$. In ⁽⁵⁾ it was shown that, as $s \rightarrow \infty$ and $q = s$, an analogous asymptotic formula for $t = n$, generally speaking, does not hold.

We shall henceforth consider the congruence

$$\sum_{j=1}^{\tau} \sum_{r_j=1}^{\alpha_j} a_{jr_j} x_{jr_j}^{n_j} \equiv y \pmod{p^s}, \quad (3)$$

where $a_{jr_j}, n_j, s \geq 3$ are integers, $(a_{jr_j}, p) = 1$, $3 \leq n_1 < \dots < n_{\tau} < p$.

Let $t = \alpha_1 + \dots + \alpha_{\tau}$, $l_j = \min(n_j, s)$; we shall call the number N , defined as follows, the harmonic index of congruence (3):

$$N = t \left(\sum_{j=1}^{\tau} \frac{\alpha_j}{l_j} \right)^{-1}.$$

For congruence (1), $N = \min(n, s)$. Denote by $T'_q(N)$ the number of solutions of congruence (3) for which inequalities (2) hold and $m \leq y \leq m + q - 1$, where m and q are integers, $0 \leq m < m + q \leq p^s$.

Theorem 1. Let $t \geq N$, and let $\varepsilon > 0$ be a real number, $1 \leq q \leq p^s$,

$$p^{s(1/2+1/l_j+\varepsilon)} \leq Q_{jr_j} \leq p^s, \quad r_j = 1, \dots, \alpha_j, \quad j = 1, \dots, \tau. \quad (4)$$

Then for the quantity $T'_q(N)$ the following expression holds:

$$T'_q(N) = qQ_1 \dots Q_t p^{-s} \{1 + O(\gamma(q)\eta(t) \max(p^{-t/2+1}, p^{-l_1(t/N-1)})\}.$$

where

$$\gamma(q) = \begin{cases} 1, & \text{if } q = 1, \\ q^{-1} \ln q, & \text{if } q \geq 2; \end{cases} \quad \eta(t) = \begin{cases} s, & \text{if } t = N, \\ 1, & \text{if } t > N. \end{cases}$$

The constant in the symbol O depends only on $n_1, \dots, n_{\tau}, \alpha_1, \dots, \alpha_{\tau}$ and ε .

The proof of the theorem is based on the following lemmas.

Lemma 1. Let $f(x_1, \dots, x_t)$ be an integral rational function with integer coefficients in t variables $x_1, \dots, x_t$, $t \geq 1$; let T be the number of solutions of the congruence

$$f(x_1, \dots, x_t) \equiv y \pmod{p^s},$$

for which the inequalities (2) hold, $m \leq y \leq m + q - 1$.

Then

$$T = qQ_1 \dots Q_t p^{-s} + \\ + O \left(\gamma_0(q) p^{-s} \sum_{k=1}^s p^k \max_{z \neq 0 \pmod{p}} \left| \sum_{x_1=M_1}^{M_1+Q_1-1} \dots \sum_{x_t=M_t}^{M_t+Q_t-1} \exp \frac{2\pi i}{p^k} z f(x_1, \dots, x_t) \right| \right)$$

with an absolute constant in the symbol O , $z \in [1, p^s - 1]$; $\gamma_0(q) = 1$, if $q = 1$, $\gamma_0(q) = \ln q$, if $q \geq 2$.

Proof. We express the number of solutions T through a trigonometric sum (see (6), question 1, a to Chapter IV)

$$T = p^{-s} \sum_{k=0}^s \sum_{\substack{z=1 \\ (z,p)=1}}^{p^k} \sum_{x_1=M_1}^{M_1+Q_1-1} \cdots \sum_{x_t=M_t}^{M_t+Q_t-1} \sum_{y=m}^{m+q-1} \exp \frac{2\pi i}{p^k} z(f(x_1, \dots, x_t) - y).$$

Hence

$$\begin{aligned} & |T - qQ_1 \cdots Q_t p^{-s}| \leq \\ & \leq p^{-s} \sum_{k=1}^s R(k) \max_{z \not\equiv 0 \pmod{p}} \left| \sum_{x_1=M_1}^{M_1+Q_1-1} \cdots \sum_{x_t=M_t}^{M_t+Q_t-1} \exp \frac{2\pi i}{p^k} z f(x_1, \dots, x_t) \right|, \quad (5) \end{aligned}$$

where

$$R(k) = \sum_{\substack{z=1 \\ (z,p)=1}}^{p^k} \left| \sum_{y=m}^{m+q-1} \exp \frac{2\pi i}{p^k} (-zy) \right| \leq \sum_{z=1}^{p^k-1} \frac{|\sin \pi z q / p^k|}{\sin \pi z / p^k}. \quad (6)$$

If $q = 1$, then $R(k) < p^k$. For $q \geq 2$, to estimate $R(k)$ we use the following result of B. I. Golubov ⁽⁷⁾, putting $M = p^k$.

There exists an absolute constant $c_0 > 0$ such that

$$\sum_{z=1}^{M-1} \frac{|\sin \pi z q / M|}{\sin \pi z / M} \leq c_0 M \ln q. \quad (7)$$

From formula (5) and inequalities (6) and (7) the assertion of the lemma follows.

Lemma 2 (Hua Loo-keng ⁽⁸⁾). Let a, q, P be integers, $(a, q) = 1$, $q \geq 1$, $P \geq 1$, $n \geq 2$.

Then for any $\varepsilon > 0$

$$\sum_{x=1}^P \exp \frac{2\pi i}{q} a x^n = \frac{P}{q} \sum_{x=1}^q \exp \frac{2\pi i}{q} a x^n + O(q^{1/2+\varepsilon}),$$

where the constant in the symbol O depends only on n and ε .

Lemma 3. Let $f(x) = a_1x + \dots + a_nx^n$ be a polynomial of degree $n \geq 2$ with integer coefficients, $p > n$ a prime, $a_n \not\equiv 0 \pmod{p}$, $s \geq 1$.

Then the estimate holds

$$\left| \sum_{x=1}^{p^s} \exp \frac{2\pi i}{p^s} f(x) \right| \leq \begin{cases} (n-1)p^{1/2}, & \text{if } s = 1, \\ (n-1)p^{s-1}, & \text{if } 2 \leq s \leq n, \\ c(n)p^{s(1-1/n)}, & \text{for any } s \geq 1. \end{cases}$$

The proof of the first of the inequalities written above is the subject of the papers ^(9,10) and § 2 of A. G. Postnikov's book ⁽¹¹⁾. The second and third inequalities are proved, respectively, in the papers of A. A. Karatsuba ⁽¹²⁾ and Hua Loo-keng ⁽¹³⁾, pp.7—12). In the case when $f(x) = ax^n$, $(a, p) = 1$, Lemma 3 follows from lemmas of I. M. Vinogradov ⁽¹⁴⁾, pp.269—271).

Proof of Theorem 1. By Lemma 1 we have

$$T'_q(N) = qQ_1 \dots Q_t p^{-s} + O \left(\gamma_0(q) p^{-s} \sum_{k=1}^s p^{kS}(k) \right), \quad (8)$$

where

$$S(k) = \prod_{j=1}^{\tau} \sum_{r_j=1}^{a_j} S_{jr_j}(k), \quad S_{jr_j}(k) = \max_{z \not\equiv 0 \pmod{p}} \left| \sum_{x_{jr_j}=M_{jr_j}}^{M_{jr_j}+Q_{jr_j}-1} \exp \frac{2\pi i}{p^k} z a_{jr_j} x_{jr_j}^{n_j} \right|.$$

We estimate the expression $S_{jr_j}(k)$ by Lemma 2:

$$S_{jr_j}(k) = \max_{z \not\equiv 0 \pmod{p}} \frac{Q_{jr_j}}{p^k} \left| \sum_{x_{jr_j}=1}^{p^k} \exp \frac{2\pi i}{p^k} z a_{jr_j} x_{jr_j}^{n_j} \right| + O(p^{k(1/2+\varepsilon)}).$$

Estimating the modulus of the complete sum on the right-hand side of the last equality by Lemma 3, depending on the value of k we obtain

$$S_{jr_j}(k) \ll \begin{cases} Q_{jr_j} p^{-1/2}, & \text{if } k = 1, \\ Q_{jr_j} p^{-1}, & \text{if } 2 \leq k \leq n_j, \\ Q_{jr_j} p^{-k/n_j}, & \text{if } n_j + 1 \leq k \leq s, \end{cases}$$

in view of the fact that Q_{jr_j} satisfies inequality (4). In estimating the remainder term in formula (8), we shall split the sum over k in accordance with the

estimates for $S_{j r_j}(k)$ obtained above. Depending on the value of s , we consider the following three cases.

1°. If $s \leq n_1$, then

$$\sum_{k=1}^s p^{kS}(k) \ll Q_1 \dots Q_t \left(p^{-t/2+1} + \sum_{k=2}^s p^{k-t} \right) \ll Q_1 \dots Q_t \max(p^{-t/2+1}, p^{-s(t/N-1)}). \quad (9)$$

2°. Let $n_j < s \leq n_{j+1}$, where j is one of the numbers $1, \dots, \tau - 1$ ($\tau \geq 2$). Then

$$\begin{aligned} \sum_{k=1}^s p^{kS}(k) &\ll Q_1 \dots Q_t \left(p^{-t/2+1} + \sum_{k=2}^{n_1} p^{k-t} + \right. \\ &\quad \left. + \sum_{\nu=1}^{j-1} \sum_{k=n_\nu+1}^{n_{\nu+1}} p^{k-(a_{\nu+1}+\dots+a_\tau)-k(a_1/n_1+\dots+a_\nu/n_\nu)} + \right. \\ &\quad \left. + \sum_{k=n_j+1}^s p^{k-(a_{j+1}+\dots+a_\tau)-k(a_1/n_1+\dots+a_j/n_j)} \right) \ll Q_1 \dots Q_t \max(p^{-t/2+1}, p^{-n_1(t/N-1)}). \end{aligned} \quad (10)$$

3°. If $s \geq n_\tau + 1$, then

$$\begin{aligned} \sum_{k=1}^s p^{kS}(k) &\ll Q_1 \dots Q_t \left(p^{-t/2+1} + \sum_{k=2}^{n_1} p^{k-t} + \right. \\ &\quad \left. + \sum_{\nu=1}^{\tau-1} \sum_{k=n_\nu+1}^{n_{\nu+1}} p^{k-(a_{\nu+1}+\dots+a_\tau)-k(a_1/n_1+\dots+a_\nu/n_\nu)} + \sum_{k=n_\tau+1}^s p^{-k(t/N-1)} \right) \ll \\ &\ll \eta(t) Q_1 \dots Q_t \max(p^{-t/2+1}, p^{-n_1(t/N-1)}), \end{aligned} \quad (11)$$

where $\eta(t) = s$ for $t = N$. If $t > N$, then $tN^{-1} - 1 \geq N_0^{-1}$, where N_0 is the least common multiple of $n_1, \dots, n_t$; consequently,

$$\sum_{k=n_\tau+1}^s p^{-k(t/N-1)} < p^{-n_1(t/N-1)} (p^{t/N-1} - 1)^{-1} \ll p^{-n_1(t/N-1)}.$$

Thus, for $t > N$, $\eta(t) = 1$. From formula (8) and inequalities (9), (10), and (11), with $\gamma(q) = q^{-1}\gamma_0(q)$, the theorem being proved follows.

From comparison of A. A. Karatsuba's result (2) and Theorem 1 we obtain

Corollary. If $n \geq 20$, $t \geq cn$, where c is an absolute constant, then the asymptotic formula holds

$$\sum_{\nu=0}^{\infty} p^{-\nu t} \sum_{\substack{z=1 \\ (z,p)=1}}^{p^{\nu}} \left(\sum_{x=1}^{p^{\nu}} \exp \frac{2\pi i}{p^{\nu}} z x^n \right)^t \exp \left(-\frac{2\pi i}{p^{\nu}} z d \right) = 1 + O(p^{-t/2+1}),$$

where the constant in the O -symbol depends only on n and t .

Let $f_j(x_j) = a_{1j}x_j + \dots + a_{n_jj}x_j^{n_j}$ be a polynomial of degree $n_j \geq 2$ with integral coefficients, $(a_{n_jj}, p) = 1$, $j = 1, \dots, t$; $p > \max(n_1, \dots, n_t)$, $s \geq 2$. Denote by $T_q(N)$ the number of solutions of the congruence

$$f_1(x_1) + \dots + f_t(x_t) \equiv y \pmod{p^s}, \quad (12)$$

when $x_1, \dots, x_t$ run through a complete system of residues modulo p^s , $m \leq y \leq m + q - 1$.

Theorem 2. If $t \geq N$, where N is the harmonic index of congruence (12), $1 \leq q \leq p^s$, then for $T_q(N)$ one has the expression

$$T_q(N) = qp^{s(t-1)} \left\{ 1 + O(\gamma(q)\eta(t) \max(p^{-t/2+1}, p^{-(t/N-1)\min(n_1, \dots, n_t, s)}) \right\}.$$

The constant in the O -symbol depends only on $n_1, \dots, n_t$; the values $\gamma(q)$ and $\eta(t)$ are determined by the conditions of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.

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