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Abstract

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MATHEMATICS

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ON THE CARDINALITY OF EXTREMALLY DISCONNECTED SPACES AND COMPLETE BOOLEAN ALGEBRAS

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It is well known that every cardinal is the cardinality of some bicom pactum. This situation changes if we pass to subcategories of bicom pacta. For example, as was shown in ⁽²⁾ under the assumption (GCH), the cardinality of every infinite dyadic bicom pactum either is a power of two or is the sum of a countable number of smaller cardinals. Here it is proved that under the assumption (GCH) the cardinality of every infinite extremally disconnected bicom pactum either is a power of two or is unattainable*. The principal method of proof of this theorem is a further development of the authors' ideas ^(1, 3) on embedding free Boolean algebras into complete Boolean algebras. All the notation and concepts occurring here may be found in ⁽¹⁻³⁾.

1. By $\mathbf{F}$ we denote the contravariant functor acting from the category of zero-dimensional bicom pacta to the category of Boolean algebras. If X is a zero-dimensional bicom pactum, then $\mathbf{F}(X)$ is the Boolean algebra of its open-closed subsets; conversely, if $\mathfrak{X}$ is a Boolean algebra, then $\mathbf{F}^{-1}(\mathfrak{X})$ is the Stone space of the algebra $\mathfrak{X}$. We transfer the action of the functor $\mathbf{F}$ in the natural way to cardinal invariants of topological spaces. For example, if wX is the topological weight of the space X , then $\mathbf{F}(wX) = \text{card } \mathbf{F}(X)$ is the cardinality of the Boolean algebra $\mathfrak{X} = \mathbf{F}(X)$; if cX is the Suslin number of the space X , then $\mathbf{F}(cX) = t\mathbf{F}(X)$ is the type of the Boolean algebra; finally, if $\chi(x, X)$ is the character of the point x in the space X , then $\mathbf{F}(\chi(x, X)) = \chi(\mathfrak{F}, \mathfrak{X})$ is the character of the ultrafilter $\mathfrak{F}$ in the algebra $\mathfrak{X}$, i.e. the minimum of the cardinalities of bases of the ultrafilter $\mathfrak{F}$ in $\mathfrak{X}$. Further, let $t\mathfrak{X}$ be the weight of the complete Boolean algebra $\mathfrak{X}$, i.e. $\min\{\text{card } A, A \subset \mathfrak{X}, \bar{A} = \mathfrak{X}\}$, where the bar denotes the closure of the set A in the (o) -topology of $\mathfrak{X}$. This means that A completely generates $\mathfrak{X}$, or, what is the same thing, $\mathfrak{X}$ is the smallest regular subalgebra of $\mathfrak{X}$ containing A . Let X be an extremally disconnected bicom pactum; then the cardinal invariant $vX = \mathbf{F}^{-1}(t\mathbf{F}(X))$ will be called the algebraic weight of X . We note that there exist extremally disconnected bicom pacta of

arbitrarily high topological weight having countable algebraic weight (⁴, ⁵).

2. Let $\mathfrak{X}$ be a complete infinite Boolean algebra. We note that $t\mathfrak{X}$ is the least among those cardinal numbers $\mathfrak{m}$ which possess the following property: every nonempty set $E \subset \mathfrak{X}$ contains a subset $E' \subset E$ with the same bounds and such that $\text{card } E' \leq \mathfrak{m}$. Consider an arbitrary set T of cardinality $t\mathfrak{X}$ and the upward directed system A of all its finite subsets. It is clear that $\text{card } A = t\mathfrak{X}$. As is known (¹), the sets of the algebra $\mathfrak{X}$ that are closed in the (o) -topology are precisely those and only those sets which contain the limits of all possible (o) -convergent generalized sequences of their elements.

Lemma 1. *Let Γ be a directed set, $\{x_\gamma, \gamma \in \Gamma\}$ a generalized sequence (o) -converging to some x . Then there exists a generalized sequence $\{x_\gamma, \alpha \in A\}$ with the same (o) -limit.*

Proof. Put

$$U_\gamma = \bigvee_{\delta > \gamma} |x - x_\delta|.$$

It is clear that $U_\gamma \downarrow 0$.

Choose in Γ a subset Γ' of cardinality not exceeding $t\mathfrak{X}$, for which

* A cardinal is called (weakly) unattainable if it is a limit cardinal and regular. (GCH) is the generalized continuum hypothesis.

will be $\bigwedge_{\gamma \in \Gamma'} U_\gamma = 0$. Let φ be some mapping of T onto Γ' . For each $a \in A$ there is an index γ_a such that $\gamma_a > \varphi(t)$ for every $t \in a$. Next put $\widetilde{U}_a = U_{\gamma_a}$, if $a \in A$. The inequalities

$$\widetilde{U}_a \leq \bigwedge_{t \in a} U_{\varphi(t)}$$

show that $\widetilde{U}_a \xrightarrow{(0)} 0$. Further, since $|x - x_a| \leq \widetilde{U}_a$, the sequence $\{x_a, a \in A\}$ (o) -converges to x . The lemma is proved.

Lemma 2. Let $E \subset X$. Then

$$\text{card } \overline{E} \leq (\text{card } E)^{tX}.$$

Proof. Denote by ζ the least cardinal whose cardinality is greater than tX . Form a transfinite sequence $\{E_\xi, \xi < \zeta\}$ of sets, in which $E_0 = E$ and each E_ξ consists of all possible (o) -limits of generalized sequences of the form $\{x_a, a \in A\}$, formed from elements of the set $\bigcup_{\eta < \xi} E_\eta$. Whatever the generalized sequence $\Lambda = \{x_a, a \in A\}$, consisting of elements of the set

$$\mathcal{E} = \bigcup_{\xi < \zeta} E_\xi,$$

by Lemma 1 the set of its terms has cardinality not exceeding tX , and, consequently,

$$\Lambda \subset \bigcup_{\xi < \xi_0} E_\xi$$

for some $\xi_0 < \zeta$, since ζ is a regular ordinal. Thus $\overline{E} = \mathcal{E}$. Estimate the cardinality of $\mathcal{E}$. Let $\mathfrak{n} = \text{card } E$, $t = tX$, and let t^+ be the cardinal following t . Then $\text{card } E_\xi \leq \mathfrak{n}^t$ for every $\xi < \zeta$. Indeed, this inequality is true for $\xi = 0$; if it is true for all $\xi < \xi_0 < \zeta$, then

$$\text{card} \left(\bigcup_{\eta < \xi_0} E_\eta \right) \leq t\mathfrak{n}^t \leq \mathfrak{n}^t \mathfrak{n}^t = \mathfrak{n}^t$$

and next

$$\text{card } E_{\xi_0} \leq \left[\text{card} \left(\bigcup_{\eta < \xi_0} E_\eta \right) \right]^A \leq (\mathfrak{n}^t)^t = \mathfrak{n}^{t^2} = \mathfrak{n}^t.$$

Now it is clear that

$$\text{card } \overline{E} \leq \text{card } \mathcal{E} \leq t^+ \mathfrak{n}^t \leq 2^t \mathfrak{n}^t \leq \mathfrak{n}^t \mathfrak{n}^t = \mathfrak{n}^t.$$

The lemma is proved.

Lemma 3. Let X be a complete Boolean algebra; let τ^* be the supremum of the weights of the homogeneous components of this algebra; let $t = tX$ be its type. Then there exist:

a) an independent system of elements T , and b) a disjoint system D , such that $\text{card } T = \tau^*$, $\text{card } D \leq t$, and $X = \overline{X}(T, D)$. In other words, the union $T \cup D$ topologically generates the algebra X .

Proof. Form a decomposition of X into disjoint components X_α , homogeneous in weight, having weights τ_α , $\alpha < \zeta$. Here ζ is some ordinal whose cardinality does not exceed t . It is clear that $\tau^* = \sup \tau_\alpha$. We may assume that the weights increase with the index: $\tau_\beta \geq \tau_\alpha$ for $\beta \geq \alpha$. Denote by E_α the set of all ordinals whose cardinalities are strictly less than τ_α . Thus $E_1 \subset E_2 \subset \dots$ and $\text{card } E_\alpha = \tau_\alpha$. For each pair of indices (α, β) , $\alpha \geq \beta$, construct a mapping $\varphi_{\alpha\beta}$ from E_α into E_β , putting $\varphi_{\alpha\beta}(\eta) = \eta$ for $\eta \in E_\beta$ and $\varphi_{\alpha\beta}(\eta) = 1$ for $\eta \in E_\alpha - E_\beta$. We have, obviously, for $\alpha \geq \beta \geq \gamma$: $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \varphi_{\alpha\beta}$, and for every α , $\varphi_{\alpha\alpha}$ is the identity mapping of E_α onto itself. In each of the components X_α there is a completely generating independent system of elements of cardinality τ_α ((1), p. 262, Theorem 2). We have in mind here relative independence of these systems, that is, independence with respect to the corresponding component X_α , regarded as an independent Boolean algebra. Let θ_α be some bijection, fixed from now on, of E_α onto T_α . Form a set T , including in it all sums of the form $\bigvee x_\alpha$, $\alpha < \zeta$, where $x_\alpha \in T_\alpha$ and, for $\alpha \geq \beta$, let

$$x_\beta = \theta_\beta \varphi_{\alpha\beta} \theta_\alpha^{-1}(x_\alpha).$$

We shall show that T is an independent set of elements. Take a finite set $x^1, x^2, \dots, x^m \in T$, whose elements are distinct. Let $X^i = \bigvee x_\alpha^i$, $\alpha < \zeta$. Then for any pair of indices $i, j \leq m$ there is $\alpha < \zeta$ such that $x_\alpha^i \neq x_\alpha^j$. Then also for all $\alpha' > \alpha$ one will have $x_{\alpha'}^i \neq x_{\alpha'}^j$ (otherwise

$$x_\alpha^i = \theta_\alpha \varphi_{\alpha' \alpha} \theta_{\alpha'}^{-1}(x_{\alpha'}^i) = \theta_\alpha \varphi_{\alpha' \alpha} \theta_{\alpha'}^{-1}(x_{\alpha'}^j) = x_\alpha^j$$

). It is clear now that there is $\alpha_0 < \zeta$ such that, for all $\alpha \geq \alpha_0$, any x_α^i, x_α^j ($i \neq j$) are distinct. And then, since $x_\alpha^i \in T_\alpha$, and T_α is relatively independent, for any p one has

$$x^1 \wedge \dots \wedge x^p \wedge Cx^{p+1} \wedge \dots \wedge Cx^m \geq x_{\alpha_0}^1 \wedge \dots \wedge x_{\alpha_0}^p \wedge Cx_{\alpha_0}^{p+1} \wedge \dots \wedge Cx_{\alpha_0}^m > 0.$$

We have thereby proved the independence of the system T . The disjoint ...

the system D , by putting $U_\alpha = \bigvee x_\alpha$, $x_\alpha \in \mathfrak{X}_\alpha$. Then $D = \{U_\alpha\}$. It is easy to see that $\mathfrak{X} = \mathfrak{X}\langle T, D \rangle$, $\text{card } D \leq t$. It remains to determine the cardinality of the system T . Let $y \in T_{\alpha_0}$. Putting $x_\alpha = \theta_\alpha \theta_{\alpha_0}^{-1}(y)$ for $\alpha \geq \alpha_0$ and $x_\alpha = \theta_\alpha \varphi_{\alpha \alpha_0} \theta_{\alpha_0}^{-1}(y)$ for $\alpha < \alpha_0$, we see that the element $\bigvee_{\alpha < \zeta} x_\alpha$ belongs to T . Therefore $\text{card } T \geq \tau_{\alpha_0}$ for every α_0 , and hence $\text{card } T \geq t^*$. Conversely, to an arbitrary ordinal $\gamma \in \bigcup_{\alpha < \zeta} E_\alpha$ we associate the least number $\bar{\alpha} = \alpha(\gamma)$ for which $\gamma \in E_{\bar{\alpha}}$, and then put $x_\alpha = \theta_\alpha(1)$ for $\alpha < \bar{\alpha}$ and $x_\alpha = \theta_\alpha(\gamma)$ for $\alpha \geq \bar{\alpha}$. Finally, let $x = x(\gamma) = \bigvee_{\alpha < \zeta} x_\alpha$. It is easy to see that $x \in T$ and that the system T contains no other elements distinct from elements of the form $x(\gamma)$. Thus a surjection of the set $\bigcup_{\alpha < \zeta} E_\alpha$ onto T has been constructed and $\text{card } T \leq \text{card } \bigcup_{\alpha < \zeta} E_\alpha = t^*$. The lemma is proved.

Lemma 4. Let E be a base of some proper ultrafilter $\mathfrak{F}$ of the complete Boolean algebra $\mathfrak{X}$. Then there exists a principal ideal $\mathfrak{A}$ which, considered as an element of $\mathfrak{X}$, belongs to $\mathfrak{F}$, and the set $\{e \wedge \{\mathfrak{A}\}, e \in E\}$ completely generates $\mathfrak{A}$.

Proof. We note that there exists a component $\mathfrak{X}_0 \subset \mathfrak{X}$ which is saturated by $\overline{\mathfrak{X}\langle E \rangle}$. Otherwise, by Lemma 2, p. 246 ⁽¹⁾, there is an element $z \in \mathfrak{X}$ such that for every $x \in \overline{\mathfrak{X}\langle E \rangle}$ the inequalities $z \wedge x > 0$ and $Cz \wedge x > 0$ hold. In particular, these inequalities are true for every $x \in E$. Since $\mathfrak{F}$ is an ultrafilter, either $z \in \mathfrak{F}$, or $Cz \in \mathfrak{F}$. Let $z \in \mathfrak{F}$. Then, by the definition of a base E , there exists $x \in E$ such that $0 < x < z$. But in this case $Cz \wedge x = 0$. A contradiction. Thus $\overline{\mathfrak{X}\langle E \rangle}$ saturates some component $\mathfrak{X}_0 \subset \mathfrak{X}$. Let $y = \{\mathfrak{X}_\alpha, \alpha \in A\}$ be a maximal family of pairwise disjoint components, each of which is saturated by $\overline{\mathfrak{X}\langle E \rangle}$. Then $x = \{\mathfrak{A}\} = \bigvee y$, $y \in \bigcup_{\alpha \in A} \mathfrak{X}_\alpha$, is a principal ideal of $\mathfrak{X}$, and $\overline{\mathfrak{X}\langle E \rangle}$ saturates $\mathfrak{A}$. We prove that $x \in \mathfrak{F}$. If $x \notin \mathfrak{F}$, then there exists a component $\mathfrak{X}_0 \subset Cx \in \mathfrak{F}$ which is saturated by $\overline{\mathfrak{X}\langle E \rangle}$, contrary to the maximality of η . The lemma is proved.

3. Let $f : X \rightarrow Y$ be a continuous mapping of the topological space X onto the space Y . We shall call the mapping f **thin** if $\text{int } f^{-1}(x) = \emptyset$ for every nonisolated point $x \in Y$.

Lemma 5. Let $\varphi : E \rightarrow \mathfrak{X}$ be the identical embedding of the subalgebra E into the algebra $\mathfrak{X}$ such that E completely generates $\mathfrak{X}$. Then the mapping $F^{-1}(\varphi) : F^{-1}(\mathfrak{X}) \rightarrow F^{-1}(E)$ is a thin mapping.

Proof. Suppose the contrary. Let x be some nonisolated point of $F^{-1}(E)$ such that $U = \text{int}[F^{-1}(\varphi)]^{-1}(x) \neq \emptyset$. Since $F^{-1}(x)$ is extremally disconnected, U is open-and-closed. Let V be another open-and-closed set, with $V \subset U$ and $U - V = V_0 \neq \emptyset$. Denote by $a = F(V)$ and $a_0 = F(V_0)$ the corresponding elements of the Boolean algebra $\mathfrak{X}$. We note that for every open-and-closed $W \subset F^{-1}(E)$ the symmetric difference $|W - V| \supset V_0$. This means that for every element $x \in E$ we have $|x - a| > a_0$, which contradicts the given condition ($\bar{E} = \mathfrak{X}$). The lemma is proved.

4. **Main results.** Let t be a cardinal. Denote by $\text{Ex}(t)$ the function of t which is equal to $2^t = \exp t$ if t is attainable, or to $\sum_{\sigma < t} \exp(\sigma)$ if t is unattainable. As Erdős and Tarski showed ⁽⁶⁾, if the type of the Boolean algebra $\mathfrak{X}$ is equal to t , and t is attainable, then in $\mathfrak{X}$ there exists a family of disjoint elements of cardinality t . We note that, by the results of Kantorovich–Fichtenholz–Hausdorff ⁽⁷⁾ on independence, every complete Boolean algebra $\mathfrak{X}$ containing t disjoint elements necessarily contains a free subalgebra of cardinality $\exp t$. Combining this observation with Lemmas 2 and 3, we obtain the following two main theorems.

Theorem 1. The cardinality of an infinite complete Boolean algebra $\mathfrak{X}$ satisfies the following inequality

$$\max[\text{Ex}(t), \tau^*] \leq \text{card } \mathfrak{X} \leq \min[\tau^t, (\tau^* + t)^t], \quad (*)$$

where t is the type of the algebra $\mathfrak{X}$, τ is the weight of $\mathfrak{X}$, and τ^* is the supremum of the weights of homogeneous components. Moreover, $\mathfrak{X}$ contains a free subalgebra of cardinality $\tau^* + \exp(t)$, if t is attainable, or $\tau^* + \xi$ for every $\xi < \text{Ex}(t)$, if t is unattainable.

Theorem 2. The weight of an infinite extremally disconnected bicomact space X satisfies the inequality $(*)$, where t is the Suslin number of X , τ is the algebraic weight of X , and τ^* is the supremum of the algebraic weights of the open-closed subsets of X that are homogeneous with respect to this weight. Moreover, X is continuously mapped onto a generalized Cantor discontinuum $D^{\mathfrak{m}}$, where $\mathfrak{m} = \tau^* + \exp(t)$, if t is attainable, or $\mathfrak{m} = \tau^* + \xi$ for every $\xi < \text{Ex}(t)$, if t is unattainable.

Below we give a table in which the values of the left- and right-hand sides of the inequality $(*)$ are computed as functions of t . Let us note that always $\tau^* \leq \tau$. Here t -a denotes that t is an attainable cardinal, and t -u denotes an unattainable cardinal.

t	t	$\max[\text{Ex}(t), \tau^*]$	$\min[\tau^t, (\tau^* + t)^t]$	No.
$t < \tau^*$	t -a	$\tau^* + \exp(t)$	$(\tau^*)^t$	1
$t < \tau^*$	t -u	$\tau^* + \sum_{\sigma < t} \exp(\sigma)$	$(\tau^*)^t$	2
$\tau^* \leq t$	t -a	$(\tau^*)^t = \exp(t)$	$(\tau^*)^t = \exp(t)$	3
$\tau^* \leq t$	t -u	$\sum_{\sigma < t} \exp(\sigma)$	$(\tau^*)^t = \exp(t)$	4

Theorem 3 (GCH). The cardinality of an infinite extremally disconnected bicomact space X is either equal to a power of two or is unattainable.

Proof. Let us note that for bicomact spaces one has

$$wX \leq \text{card } X \leq \exp(wX). \quad (**)$$

In cases 1) and 2), listed in the table, we have $\tau^* \leq wX \leq (\tau^*)^t$, with $t < \tau^*$, and X is mapped onto D^{τ^*} . Hence, applying (**), we obtain

$$\exp(\tau^*) \leq \text{card } X \leq \exp[(\tau^*)^t] \leq \exp \exp \tau^*.$$

Thus, under (GCH), either $\text{card } X = \exp \tau^*$ or $\text{card } X = \exp \exp \tau^*$. In case 3) one has $wX = \exp t$, and X is mapped onto $D^{\exp t}$. Hence $\text{card } X = \exp \exp t$. Finally, in case 4) we have $t \leq wX \leq \exp t$, with t unattainable. Hence, again applying (**), we obtain

$$t \leq \text{card } X \leq \exp \exp t.$$

Thus, under (GCH), the cardinality of X is equal either to t , or to $\exp t$, or to $\exp \exp t$, with t unattainable. The theorem is proved.

Theorem 4. Let X be an extremally disconnected bicomact space and let $x \in X$. Then there exists a neighborhood U of the point x such that $wU \leq [\chi(x, X)]^{c^x}$.

Theorem 5. Every extremally disconnected bicomact space homogeneous with respect to algebraic weight τ is mapped onto D^τ by means of an irreducible mapping.

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Note: Figure translations are in progress. See original paper for figures.

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