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Abstract

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MATHEMATICS

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ON BICOMPACTA WITH NONCOINCIDING INDUCTIVE DIMENSIONS

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In ⁽²⁾ a description was given of a bicom pactum X with $\dim X = \text{ind } X = 2$, $\text{Ind } X = 3$. Recently I. K. Lifanov and B. A. Pasynkov substantially simplified this example, constructing a bicom pactum X with $\dim X = \text{ind } X = 3$, $\text{Ind } X = 4^*$. In this note we shall construct a series of examples of this sort, which in one respect or another are better than those named.

§ 1. **A bicom pactum R with $\dim R = \text{ind } R = 2$, $\text{Ind } R = 3$.** In the set $C_0 = I \times I \times D$, where $I = [0, 1]$, $D = \{0, 1\}$, introduce the lexicographic order as follows: $(r_1, r_2, r_3) > (r'_1, r'_2, r'_3)$ if $r_1 > r'_1$, or $r_1 = r'_1$, $r_2 > r'_2$, or $r_1 = r'_1$, $r_2 = r'_2$, $r_3 > r'_3$. An order is introduced analogously in the product $J = I \times K$, where K is the Cantor perfect set. Both these spaces, as is easily seen, are ordered zero-dimensional bicom pacta with the first axiom of countability. In what follows we shall use the bicom pactum constructed in ⁽³⁾. We shall use the notation from ⁽³⁾. Gluing together the ends of the gaps in J , we obtain the lexicographically ordered square. This, together with the constructions of ⁽³⁾, gives a representation of the space Y_i ($i = 1, 2$) as the image of a zero-dimensional bicom pactum with the first axiom of countability: $\Phi : \theta = J \times K \rightarrow Y_i$, $i = 1, 2$. After this, in ⁽³⁾, from the disjoint sum of the one-dimensional in all senses bicom pacta Y_1 and Y_2 , after gluing by Φ , a two-dimensional in the inductive senses bicom pactum X is obtained.

Let s be a point not belonging to θ . In the product $C = C_0 \times (\{s\} \cup \theta)$ introduce a topology in the following way. The set $\{c\} \times \theta$, $c \in C_0$, will be regarded as an open-and-closed subspace in the space C with topology θ . A neighborhood of the point (c_0, s) , $c_0 \in C_0 \subset C$, will be called a set $l \times (\{s\} \cup \theta)$, where l is a neighborhood of the point c_0 in C_0 , and also any set obtained from this by throwing out a finite number of sets of the form $\{c\} \times \theta$, $c \in C_0$.

It is not difficult to construct in the square I^2 a countable base D consisting of rectangles (of the form $l_1 \times l_2$, $l_1, l_2 \subset I^2$) of diameter less than $1/4$, such that the boundaries of any two intersect in no more than four points that are not vertices of squares, and no point of the square I^2 belongs to more than two

boundaries. Let Q be the set of points of the square I^2 belonging to more than one boundary. It is, as is easy to see, countable.

Take a point $q \in Q$. It lies at the intersection of the boundaries of two squares d_0 and d_1 of the family D . Let l_i , $i = 0, 1$, be the ray emanating from the point q , passing along the boundary of the square d_i . Let t_i , $i = 0, 1$, be the part lying in the square I^2 of the closed sector with vertex at q , containing the ray l_i and bounded by rays forming angles $\pm 30^\circ$ with l_i . The intersection of these sectors, as is easily seen, consists only of their common vertex q . The triple (q, t_0, t_1) will be called marked. Denote the set of marked triples by Σ_1 .

A pair (x, N) , $x \in I^2$, $N \subseteq I^2$, will be called marked if: a) $N = I^2$, if the point x does not belong to the boundary of any square of the base D ; b) $N = [d]$

* The bicom pactum T^3 from ⁽⁴⁾.

or $I^2 \setminus d$, if the point $x \notin Q$ lies on the boundary of a square $d \in D$; c) $N = [d_1] \cap [d_2]$, or $[d_1] \cap (I^2 \setminus d_2)$, or $(I^2 \setminus d_1) \cap [d_2]$, or $(I^2 \setminus d_1) \cap (I^2 \setminus d_2)$, where d_1, d_2 are distinct elements of the base D , in the intersection of whose boundaries the point x lies, if $x \in Q$. We denote the set of marked pairs by Σ_2 . As is easy to see, the set $\Sigma = \Sigma_1 \cup \Sigma_2$ has the same cardinality as the interval I . Let $f: I \rightarrow \Sigma$ be a one-to-one correspondence.

In the product $I^2 \times C$, endowed with the Tychonoff topology, we perform the following rearrangements: a) if $f(r_2)$ is a marked pair (x, N) , then from the space we remove the sets $(I^2 \setminus N) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, and in the sets $\{x\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, we perform the decompositions ψ_i , and then glue these spaces $\psi_i(\theta)$, $i = 0, 1$, by Φ and obtain a space homeomorphic to X ; b) if $f(r_2)$ is a marked triple (q, t_0, t_1) , then from the space we remove the sets $(I^2 \setminus t_i) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, and in the sets $\{q\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, we perform the decompositions ψ_i , and then these spaces $\psi_i(\theta)$, $i = 0, 1$, are glued by Φ , and we obtain a space homeomorphic to X . What is obtained under these rearrangements from a set $M \subset I^2 \times C$ will be denoted by $\Psi(M)$. Let $\Psi(I^2 \times C) = R$.

The dimension estimates for the space R are obtained as follows. Since R contains sets homeomorphic to a square, all dimensions of R are not less than 2. By tracing the images under Ψ of cubic neighborhoods in $I^2 \times C$ and their finite unions, it is not difficult to verify that all dimensions are not greater than 3.

We show that $\text{ind } R \leq 2$. For points of $\Psi((r_1, r_2, i) \times \theta)$, $i = 0, 1$, this estimate is obvious. As is easy to see, the set $\text{Int } \Psi(l \times [d])$, where $d \in D$, l is a clopen set in C , has a one-dimensional boundary. Altogether this gives the required estimate.

We show that $\text{Ind } R \geq 3$. By standard arguments (see (3)) one can show that any partition between the sets $\Psi(\{0\} \times I \times C)$ and $\Psi(\{1\} \times I \times C)$ contains either a set open in some subspace of the form $\Psi(I^2 \times \{(r_1, r_2, i)\})$, $i = 0$ or 1 , or some set of the form $[\Psi(E \times (\{r_1\} \times I \times D \times (\{1\} \cup \theta))) \cap R^0]$, where R^0 is

the set of points of R at which the open mapping $\pi' : R \rightarrow I^2$, generated by the projection onto the factor $\pi : I^2 \times C \rightarrow I^2$, is open, and E is a partition between $\{0\} \times I$ and $\{1\} \times I$ in I^2 . As is easy to see, in both cases the dimension of the boundary is not less than 2, and therefore $\text{Ind } R \geq 3$.

§ 2. A bicompactum R_2 with $\dim R_2 = 1$, $\text{ind } R_2 = 2$, $\text{Ind } R_2 = 3$. In the Sierpiński carpet S , obtained from the square I^2 by removing a countable family Δ of open squares (see (1)), one can choose a countable base D , consisting of intersections with S of rectangles lying in I^2 of diameter less than $1/4$, such that: a) the boundaries of any two rectangles intersect in at most four points that are not their vertices; b) no point of the carpet S belongs to the boundaries of more than two elements of the base D ; c) the boundaries of the elements of the base D are connected; d) the boundary of an element of the base D intersects the boundary of a square of the family Δ in two points or does not intersect it at all.

Let Q be the set of points of the carpet S belonging to the boundaries of more than one element of the family D . It is, as is easy to see, countable.

To each element δ of the family Δ we assign a copy $X(\delta)$ of the space X from (3). On the set

$$S^* = S \cup \left(\bigcup_{\delta \in \Delta} X(\delta) \right)$$

we introduce a topology as follows. The subspaces $X(\delta)$ will be considered clopen. A neighborhood of a point from $S \subset S^*$ will be a set obtained from an element d of the family D by adjoining those sets $X(\delta)$ for which the boundary of the square δ lies entirely in d .

Let the symbols $\psi_i, \theta, Y_i, \Phi, X, C$ denote the same as in § 1. By marked pairs (x, N) and marked triples (q, t_0, t_1) we shall mean pairs and triples described in the same way as in § 1, with the difference that, instead of N, t_0, t_1 from § 1, their intersections with S are taken, with the addition of those sets ...

sets $X(\delta)$ for which the boundary of the square δ lies entirely in this intersection. Let $f : I \rightarrow \Sigma$ be a one-to-one correspondence between the points of the segment and the set Σ of all marked pairs and triples.

In the product $S^* \times C$, taken in the Tikhonov topology, we make the following rearrangements: a) if $f(r_2)$ is a marked pair (x, N) , then we throw out from the space the sets $(S^* \setminus N) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, in the sets $\{x\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, perform the decompositions ψ_i , and then glue these spaces $\psi_i(\theta)$, $i = 0, 1$, by Φ and obtain a space homeomorphic to X ; b) if $f(r_2)$ is a marked triple (q, t_0, t_1) , then we throw out from the space the sets $(I^2 \setminus t_i) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, in the sets $\{q\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, perform the decompositions ψ_i , and then glue these spaces $\psi_i(\theta)$, $i = 0, 1$, by Φ and obtain a space homeomorphic to X .

The estimates of the dimensions of the bicompactum R_3 obtained after these

rearrangements are made in the same way as in § 1: $\dim R_2 = 1$, $\text{ind } R_2 = 2$, $\text{Ind } R_2 = 3$.

§ 3. Construction, from a bicom pactum T with $\dim T = l$, $\text{ind } T = m$, $\text{Ind } T = n$, $1 \leq l \leq m \leq n$, of a bicom pactum T^* with $\dim T^* = l$, $\text{ind } T^* = m + 1$, $\text{Ind } T^* = n + 1$, where $T^* = T_1 \cup T_2$, with T_i , $i = 1, 2$, bicom pacts satisfying $\dim T_i = l$, $\text{ind } T_i = m$, $\text{Ind } T_i = n$.

In the square I^2 one can find a vertical $\{\vartheta\} \times I$ which does not intersect the closures of the elements of the family Δ of squares thrown out of the square I^2 in the construction of the Sierpiński carpet S . To each element δ of the family Δ we assign a copy $T(\delta)$ of the bicom pactum T . On the set

$$S^* = S \cup \left(\bigcup_{\delta \in \Delta} T(\delta) \right)$$

we introduce a topology in the same way as in the analogous situation in § 2.

Let $\varphi_i : \theta \rightarrow T$ be a mapping of some zero-dimensional bicom pactum θ onto the bicom pactum T such that for every point $t \in T$ (respectively, every closed set $T' \subseteq T$) there is an arbitrarily small neighborhood V with $(m-1)$ -dimensional (respectively, $(n-1)$ -dimensional) boundary such that the set $\varphi^{-1}([V])$ is open. At least one such bicom pactum θ exists, for example, the absolute of the bicom pactum T .

Let, as before, $C_0 = I \times I \times D$ carry the lexicographic order. In the set $C = C_0 \cup (I \times I \times \theta)$ we introduce a topology in the following way. The set $\{r_1\} \times \{r_2\} \times \theta$ will be regarded as an open-and-closed subspace with topology θ . Convergence to points from C_0 is interval convergence, taking into account that the set $\{r_1\} \times \{r_2\} \times \theta$ lies between the points $(r_1, r_2, 0)$ and $(r_1, r_2, 1)$.

We shall call marked a pair (x, N) , where $x = (\vartheta, r) \in S$, and N is obtained from one of the sets

$$N_0 = ([0, \vartheta] \times [0, r] \cup [\vartheta, 1] \times [r, 1]) \cap S$$

or

$$N_1 = ([0, \vartheta] \times [r, 1] \cup [\vartheta, 1] \times [0, r]) \cap S$$

by adding those sets $T(\delta)$ for which the boundary of the square δ intersects the corresponding set N_i .

Let $f : I \rightarrow \Sigma$ be a one-to-one correspondence between the points of the segment and the set Σ of all marked pairs.

In the Tikhonov product $S^* \times C$ we make the following rearrangements: if $f(r_2) = (x, N)$, then we throw out from the space the set $(S^* \setminus N) \times (\{r_1\} \times \{r_2\} \times \theta)$, and in the set $\{x\} \times (\{r_1\} \times \{r_2\} \times \theta)$ we perform the identification φ . The estimates of the dimensions of the bicom pactum T^* obtained as a result of these rearrangements Φ and of its closed subspaces $T_i = \Phi(S_i^* \times C)$, $i = 1, 2$,

where S_1^* and S_2^* are the sets of points of the bicom pactum S^* lying respectively not to the right and not to the left of the vertical $\{\vartheta\} \times \Gamma$, are carried out as in § 1.

§ 4. Construction, from a bicom pactum T^* with $\dim T^* = l$, $\text{ind } T^* = m + 1$, $\text{Ind } T^* = n + 1$, representable as the union of two of its closed subsets $T^* = T_1 \cup T_2$ with $\dim T_i = l$, $\text{ind } T_i = m$, $\text{Ind } T_i = n$, $i = 1, 2$, of a bicom pactum T^{**} with $\dim T^{**} = l$, $\text{ind } T^{**} = m + 1$, $\text{Ind } T^{**} = n + 2$.

Let $\varphi : \theta \rightarrow T$ be a continuous mapping of some zero-dimensional bicom pactum θ onto a bicom pactum T such that at each point $t \in T$ (respectively, at each closed set $T' \subseteq T$) there is an arbitrarily small neighborhood with m -dimensional (respectively n -dimensional) boundary such that the set $f^{-1}([V])$ is open. We shall also assume that $\theta_1 \cap \theta_2 = \emptyset$, where $\theta_i = \varphi^{-1}(T_i)$, $i = 1, 2$.

Let C denote (taking into account the change in the meaning of θ) the same thing as in § 3. Let D be the base in the Sierpiński carpet S described in § 2. Let the bicom pactum S^* be obtained from the carpet S in the same way as in § 2, with the only difference that, instead of the set $X(\delta)$, we shall add copies $T^*(\delta)$ of the bicom pactum T^* . The set Q , the marked pairs, triples, and the correspondence $f : I \rightarrow \Sigma$ are the same as in § 2, taking into account the difference in the construction.

In the product $S^* \times C$, taken in the Tikhonov topology, we make the following rearrangements: a) if $f(r_2)$ is a marked pair (x, N) , then we remove from the space the sets $(S^* \setminus N) \times (\{r_1\} \times \{r_2\} \times \theta)$ and in the set $\{x\} \times (\{r_1\} \times \{r_2\} \times 0)$ make the identification φ ; b) if $f(r_2)$ is a marked triple (q, t_0, t_1) , then we remove from the space the sets $(I^2 \setminus t_i) \times (\{r_1\} \times \{r_2\} \times \theta_i)$, $i = 1, 2$, and in the set $\{x\} \times (\{r_1\} \times \{r_2\} \times \theta)$ carry out the identification φ .

The estimates of the dimensions of the bicom pactum T^{**} obtained as a result of these rearrangements are made in the same way as in the preceding cases.

§ 5. A bicom pactum R_i with $\dim R_i = 1$, $\text{ind } R_i = i$, $\text{Ind } R_i = 2i - 1$. As the bicom pactum R_1 we shall take the interval I . Having first carried out with it (as T) the constructions of § 3, and then the constructions of § 4, we obtain a bicom pactum T^{**} with $\dim T^{**} = 1$, $\text{ind } T^{**} = 2$, $\text{Ind } T^{**} = 3$. We shall take this bicom pactum as R_2 . Continuing this process, we obtain bicom pacta R_i for all i . As is easy to see (this is proved by induction), starting with the interval, we can arrange that in all the bicom pacta the first axiom of countability is satisfied.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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