

Topological Structure of Normally Solvable Elliptic Operators with Constant Coefficients in the Plane

P. A. Frolov

1970

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-197001.74045>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1970. Volume 191, No. 6

UDC 517.94

MATHEMATICS

P. A. FROLOV

TOPOLOGICAL STRUCTURE OF NORMALLY SOLVABLE ELLIPTIC OPERATORS WITH CONSTANT COEFFICIENTS IN THE PLANE

(Presented by Academician A. Yu. Ishlinskii on 30 VI 1969)

The differential operator

$$I = \sum_{j=0}^n A_j \frac{\partial^n}{\partial x^{n-j} \partial y^j} + \sum_{k+l < n} B_{kl} \frac{\partial^{k+l}}{\partial x^k \partial y^l}, \quad (1)$$

where A_j, B_{kl} are constant real $m \times m$ matrices, is called elliptic if the corresponding matrix pencil $L(\lambda)$ of the form

$$L(\lambda) = \lambda^n A_0 + \lambda^{n-1} A_1 + \dots + A_n \quad (2)$$

satisfies one of the conditions:

$$\det A_0 > 0, \quad \det L(\lambda) > 0, \quad -\infty < \lambda < +\infty, \quad (3)$$

or

$$\det A_0 < 0, \quad \det L(\lambda) < 0, \quad -\infty < \lambda < +\infty. \quad (3')$$

The sets of operators satisfying conditions (3) and (3') will be denoted by $\mathcal{L}_+(n, m)$ and $\mathcal{L}_-(n, m)$, respectively. As we showed in ⁽¹⁾, each of the sets (3) and (3'), for $m > 2$, consists of two connected components. The components of $\mathcal{L}_+(n, m)$ will be denoted by $l_1^+(n, m)$ and $l_2^+(n, m)$. As we showed in ⁽¹⁾, every operator from $l_1^+(n, m)$ is homotopic to the operator

$$I_{01} = \Delta^{n/2} E_m \quad \text{for even } n,$$

$$I_{11} = \Delta^q \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \quad \text{for odd } n, \quad n = 2q + 1,$$

where

$$\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2; \quad K = \begin{pmatrix} \partial / \partial x & \partial / \partial y \\ -\partial / \partial y & \partial / \partial x \end{pmatrix}.$$

We shall now indicate a necessary and sufficient condition under which the operator (1) from $\mathcal{L}_+(n, m)$ belongs to the first component $l_1^+(n, m)$. To this end, with the operator (1) we associate an $n \cdot m \times n \cdot m$ matrix $\mathfrak{A}$:

$$\mathfrak{A} = \begin{pmatrix} 0 & E & 0 & \cdots & 0 \\ 0 & 0 & E & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & E \\ -A_0^{-1}A_n & \cdots & \cdots & \cdots & -A_0^{-1}A_1 \end{pmatrix}. \quad (4)$$

The spectrum of the pencil (2) coincides with the spectrum of the matrix $\mathfrak{A}$. Let the $n \cdot m/2$ -vectors

$$f_1, f_2, \dots, f_r, \quad r = n \cdot m/2, \quad (5)$$

form a basis in the invariant subspace of the matrix $\mathfrak{A}$ corresponding to the spectrum from the upper half-plane. Consider the determinant

$$\delta_L = (i)^r \det(f_1, f_2, \dots, f_r, \bar{f}_1, \bar{f}_2, \dots, \bar{f}_r), \quad (6)$$

where $i = \sqrt{-1}$. It is not hard to prove that δ_L is real.

Theorem 1. *In order that an elliptic operator of the form (1) from $\mathcal{L}_+(n, m)$ belong to the component $l_1^+(n, m)$, it is necessary and sufficient that, for even n ($n = 2q$),*

$$\text{sign } \delta_L = (-1)^{q(q-1)m/2} \quad (7)$$

and, for odd n ($n = 2q + 1$),

$$\text{sign } \delta_L = (-1)^{m(q^2 - q - m)/2}. \quad (7')$$

Fig. 1

Figure 1: Fig. 1

We outline the proof of the theorem. Let $L_\varepsilon(\lambda)$ ($0 \leq \varepsilon \leq 1$) be a deformation of the bundle $L_0(\lambda)$ corresponding to an operator from $\mathcal{L}_+(n, m)$. The basis (5) corresponding to $L_\varepsilon(\lambda)$ can be constructed so as to depend continuously on $\varepsilon \in [0, 1]$; consequently, $\delta_{L_0} = \delta_{L_1}$. It is not hard to calculate that the δ_L corresponding to I_{01} and I_{11} are computed by the formulas (7) and (7'), and, moreover, $\delta_{L_{01}} = -\delta_{L_{02}}$, $\delta_{L_{11}} = -\delta_{L_{22}}$, where $L_{02}^{(\lambda)}, L_{22}^{(\lambda)}$ are the bundles corresponding to the canonical representatives from $l_2^+(n, m)$ for odd n (see (1)).

Fig. 1

2. Let G be a domain in the (x, y) -plane, bounded by an infinitely smooth Jordan curve Γ . Denote by $\mathcal{H}_l(G)$, where l is an integer $> n/2$, the direct product of m spaces

$$\mathcal{H}_{l-1/2}(\Gamma) \times \mathcal{H}_{l-1-1/2}(\Gamma) \times \cdots \times \mathcal{H}_{l-n/2-1/2}(\Gamma).$$

For the operator (1) of even order $n = 2q$, consider the Dirichlet problem:

$$I(u) = f, \quad f \in \mathcal{H}_{l-n}(G), \quad (8)$$

$$u|_\Gamma = g_0, \quad \partial u / \partial \nu|_\Gamma = g_1, \dots, \partial^{q-1} u / \partial \nu^{q-1}|_\Gamma = g_{q-1}, \quad (9)$$

where $\partial / \partial \nu$ is differentiation along the normal to the boundary Γ , and $g_j \in \mathcal{H}_{l-j-1/2}$.

The operator from $\mathcal{H}_l(G)$ to $\mathcal{H}_{l-g-1/2}(\Gamma)$, defined by the equalities (9), will be denoted by D .

We shall call problem (7)–(8) normally solvable if the operator $N = (I, D)$, acting from $\mathcal{H}_l(G)$ to $\mathcal{H}_{l-n}(G) \times \mathcal{H}_{l-g-1/2}(\Gamma)$, is Noetherian, i.e., the kernel and cokernel of the operator N have finite dimensions and the range of N is closed in $\mathcal{H}_{l-n}(G) \times \mathcal{H}_{l-g-1/2}(\Gamma)$ (see (2)).

Without loss of generality, we shall assume that all eigenvalues of the bundle (2) are simple. Then, as can be proved using the result of Lopatinskii (see (3)), in order that problem (8)–(9) be normally solvable, it is necessary and sufficient that

$$d = \det \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_r \\ \lambda_1 \psi_1 & \lambda_2 \psi_2 & \cdots & \lambda_r \psi_r \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{q-1} \psi_1 & \lambda_2^{q-1} \psi_2 & \cdots & \lambda_r^{q-1} \psi_r \end{pmatrix} \neq 0, \quad (10)$$

where $\lambda_1, \lambda_2, \dots, \lambda_r$ are the eigenvalues of the bundle (2) in the upper half-plane, and $\psi_1, \psi_2, \dots, \psi_r$ are the corresponding eigenvectors.

Theorem 2. *The set of elliptic operators (1) belonging to $\mathcal{L}_+(n, m)$, for which the Dirichlet problem is normally solvable, splits into two connected components: $\tilde{l}_1^+(n, m)$ and $\tilde{l}_2^+(n, m)$; moreover, $\tilde{l}_1^+(n, m)$ and $\tilde{l}_2^+(n, m)$ are open everywhere dense connected subsets of the components $l_1^+(n, m)$ and $l_2^+(n, m)$, respectively.*

We outline the proof of this theorem. Let, for definiteness, the matrix bundles $L_1(\lambda)$ and $L_2(\lambda)$ correspond to operators from the component $l_2^+(n, m)$, and let condition (10) be fulfilled for both of these bundles.

Using the representation of $L_1(\lambda)$ and $L_2(\lambda)$ as products of linear bundles (see (1)), one can prove that there exists a matrix bundle $L_\varepsilon(\lambda)$, analytically dependent on ε , $0 \leq \varepsilon \leq 1$, belonging to $l_1^+(n, m)$ for all ε :

$$L_\varepsilon(\lambda)|_{\varepsilon=0} = L_1(\lambda); \quad L_\varepsilon(\lambda)|_{\varepsilon=1} = L_2(\lambda)$$

and such that the eigenvalues $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \dots, \lambda_r(\varepsilon)$, $\text{Im } \lambda_j(\varepsilon) > 0$, and the corresponding eigenvectors $\psi_1(\varepsilon), \psi_2(\varepsilon), \dots, \psi_r(\varepsilon)$ are analytic functions in a neighborhood of the segment $[0, 1]$ of the complex ε -plane. The function $d(\varepsilon)$ (see (10)) is analytic, is not identically equal to zero, and therefore has at most a finite number of zeros on the segment $[0, 1]$. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ be the zeros of $d(\varepsilon)$ on the segment $[0, 1]$; we can bypass them in the complex ε -plane by varying ε along a contour γ connecting the points 0 and 1 (see Fig. 1), where the semicircles with centers at the zeros $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ have sufficiently small radius. We construct a bundle depending on ε , with eigenvalues $\lambda_1(\varepsilon), \bar{\lambda}_1(\varepsilon), \dots, \lambda_r(\varepsilon), \bar{\lambda}_r(\varepsilon)$ and the corresponding eigenvectors $\psi_1(\varepsilon), \bar{\psi}_1(\varepsilon), \dots, \psi_r(\varepsilon), \bar{\psi}_r(\varepsilon)$, where ε varies from 0 to 1 along the contour γ . This bundle, as is not difficult to show, is real for all $\varepsilon \in [0, 1]$ and solves the indicated problem.

We note that each of the connected components $l_1^+(n, m)$ and $l_2^+(n, m)$ also contains operators for which the Dirichlet problem is not normally solvable. For example, the operator

$$\begin{pmatrix} B_0 & 0 \\ 0 & B_0 \end{pmatrix},$$

where B_0 is the Bitsadze operator (see (*))

$$B_0 = \begin{pmatrix} \partial^2/\partial x^2 - \partial^2/\partial y^2 & 2\partial^2/\partial x \partial y \\ -2\partial^2/\partial x \partial y & \partial^2/\partial x^2 - \partial^2/\partial y^2 \end{pmatrix},$$

belongs to $l_1^+(n, m)$ and, consequently, is homotopic to the operator ΔE_4 . This assertion contains the answer to a question posed by I. M. Gelfand (see (5)).

By the same method one can prove that if the Dirichlet conditions on the boundary Γ are replaced by arbitrary conditions with coefficients independent of the point of the boundary, then the set of operators (1) from $\mathcal{L}^+(n, m)$ for which this boundary-value problem is normally solvable splits into two connected components.

The author expresses his gratitude to Prof. V. B. Lidskii for valuable advice and attention to the present work.

Moscow Institute of Physics and Technology

Received

8 IV 1969

REFERENCES

1. P. A. Frolov, DAN, **181**, No. 6 (1968).
2. M. S. Agranovich, UMN, **20**, issue 5 (1965).
3. Ya. B. Lopatinskii, L'vov, *Scientific Notes of the Polytechnic Institute*, **38**, series phys.-math., 2, 3 (1956).
4. A. V. Bitsadze, *Boundary-value problems for elliptic equations of the second order*, Moscow, 1956, p. 87.
5. I. M. Gelfand, UMN, **14**, issue 3, 6 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.