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Abstract

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MATHEMATICS

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ON THE DISCRETENESS OF THE SPECTRUM OF NON-SELF-ADJOINT OPERATORS

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In the present note we give theorems that make it possible to reduce the question of the discreteness of the spectrum of certain non-self-adjoint operators to the question of the discreteness of the spectrum of self-adjoint operators.

1. Let D_0 be a linear manifold dense in a Hilbert space H ; let L_1 and L_2 be operators symmetric on D_0 . Suppose that the operator $L'_0 = L_1 + iL_2$ has a closure L_0 . Then the following theorems are valid.

Theorem 1. *If at least one of the operators $L_1, L_2, L_1 + L_2, L_1 - L_2$ is semibounded and its semibounded self-adjoint extension has a discrete spectrum, then the kernel of the spectrum of the operator L_0 is discrete.*

Proof. Let, for example, $(L_1 y, y) \geq (y, y)$ for $y \in D_0$, and let $\tilde{L}_1$ be a semibounded self-adjoint extension of the operator L_1 . The set

$$\tilde{E}_1 = [y : (\tilde{L}_1 y, y) \leq 1, y \in D(\tilde{L}_1)]$$

is compact in H , since $\tilde{L}_1$ has a discrete spectrum. All the more compact is the set

$$E_1 = [y : (L_1 y, y) \leq 1, y \in D_0].$$

The set

$$E_0 = [y : |(L_0 y, y)| \leq 1, y \in D_0] \subset E_1,$$

and therefore it is also compact. Suppose that L_0 has a point λ_0 of the continuous spectrum; then, for any $\varepsilon > 0$, there exists a noncompact sequence $\{\varphi_i\}$, $\varphi_i \in D_0$, such that $1/2 \leq \|\varphi_i\| \leq 2$ and

$$\|L_0 \varphi_i - \lambda_0 \varphi_i\| \leq \varepsilon. \quad (1)$$

Put $\varepsilon = 1$. From the last inequality we have

$$\|L_0 \varphi_i\| \leq |\lambda_0| \|\varphi_i\| + 1 \leq 2|\lambda_0| + 1. \quad (2)$$

Obviously, the sequence $\{\varphi'_i\}$, where $\varphi'_i = \varphi_i/2(|\lambda_0| + 1)$, is noncompact together with the sequence $\{\varphi_i\}$, but, by virtue of inequality (2), it belongs to E_0 . We have obtained a contradiction to the compactness of the set E_0 . The remaining cases are easily reduced to the one proved.

The theorem just proved generalizes Theorem 4 of Ch. II of the paper ⁽⁶⁾.

Theorem 2. *If*

$$(L_1 y, y) \geq (y, y) \quad \text{for } y \in D_0; \quad (L_2 y, y) \geq (y, y) \quad \text{for } y \in D_0, \quad (3)$$

and the operator L_0 has no residual spectrum, then the operator L_0 has a discrete spectrum if and only if the set

$$E_0 = [y : |(L_0 y, y)| \leq 1, y \in D_0]$$

is compact in H .

Proof of sufficiency. Suppose the set E_0 is compact; then the set

$$E = [y : (L_0 y, y) \leq 1, y \in D(L_0)] \subset \widetilde{E}_0$$

is also compact, where $\widetilde{E}_0$ is the closure of the set E_0 in H . The set

$$E' = [y : \|L_0 y\| \leq 1, y \in D(L_0)] \subset E$$

is consequently also compact in H ; therefore the operator L_0^{-1} is completely continuous, and hence the spectrum of the operator L_0 is discrete (the existence of the operator L_0^{-1} follows from the conditions (3) and from the absence of residual spectrum for the operator L_0).

Proof of necessity. Suppose that the spectrum of the operator L_0 is discrete; then the inverse operator $L_0^{-1} = A$ is completely continuous. From

from condition (3) we obtain that $A = A_1 - iA_2$, where A_1 and A_2 are positive self-adjoint completely continuous operators.

Suppose that the set E_0 is not compact; then there is a noncompact sequence $\{y_n\} \subset E_0$. From (3) it follows that $\|y_n\| \leq 1$. Denote $L_0 y_n = \varphi_n$; then

$$(L_0 y_n, y_n) = (\varphi_n, A \varphi_n) = (A_1 \varphi_n, \varphi_n) - i(A_2 \varphi_n, \varphi_n). \quad (4)$$

Since A_1 and A_2 are positive self-adjoint completely continuous operators, there exist orthonormal systems $\{e_i\}$ and $\{g_i\}$ such that

$$A_1 \varphi_n = \sum_1^\infty \lambda_i(A_1)(\varphi_n, e_i)e_i, \quad A_2 \varphi_n = \sum_1^\infty \lambda_i(A_2)(\varphi_n, g_i)g_i, \quad (5)$$

where $\lambda_i(A_1) > 0$, $\lambda_i(A_2) > 0$ and tend to zero as $i \rightarrow \infty$. We shall prove that for every $\nu > 0$ there is an n such that at least one of the inequalities

$$\left\| \sum_{\nu}^{\infty} \lambda_i(A_1)(\varphi_n, e_i)e_i \right\| \geq \varepsilon_0, \quad \left\| \sum_{\nu}^{\infty} \lambda_i(A_2)(\varphi_n, g_i)g_i \right\| \geq \varepsilon_0, \quad (6)$$

holds, where $\varepsilon_0 > 0$ is independent of both n and ν .

Suppose that the assertion is false; then for every $\varepsilon > 0$ there is a ν such that

$$\left\| \sum_{\nu}^{\infty} \lambda_i(A_1)(\varphi_n, e_i)e_i \right\| < \varepsilon, \quad \left\| \sum_{\nu}^{\infty} \lambda_i(A_2)(\varphi_n, g_i)g_i \right\| < \varepsilon \quad (7)$$

for all n . Then

$$\begin{aligned} A\varphi_n &= A_1\varphi_n - iA_2\varphi_n \\ &= \left[\sum_1^{\nu} \lambda_i(A_1)(\varphi_n, e_i)e_i - i \sum_1^{\nu} \lambda_i(A_2)(\varphi_n, g_i)g_i \right] + \\ &\quad + \left[\sum_{\nu}^{\infty} \lambda_i(A_1)(\varphi_n, e_i)e_i - i \sum_{\nu}^{\infty} \lambda_i(A_2)(\varphi_n, g_i)g_i \right] = g_n^{\nu} + \psi_n^{\nu}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} g_n^{\nu} &= \sum_1^{\nu} \lambda_i(A_1)(\varphi_n, e_i)e_i - i \sum_1^{\nu} \lambda_i(A_2)(\varphi_n, g_i)g_i, \\ \psi_n^{\nu} &= \sum_{\nu}^{\infty} \lambda_i(A_1)(\varphi_n, e_i)e_i - i \sum_{\nu}^{\infty} \lambda_i(A_2)(\varphi_n, g_i)g_i. \end{aligned}$$

By inequalities (7) and equality (8), $\|\psi_n^{\nu}\| \leq 2\varepsilon$, and $\|g_n^{\nu}\| \leq 1 + 2\varepsilon$. The set $\{g_n^{\nu}\}$ belongs to a finite-dimensional space and is bounded; therefore there exists a finite ε -net $\{\eta_i\}$ for it, and since $\|\psi_n^{\nu}\| \leq 2\varepsilon$, the set $\{\eta_i\}$ forms a 3ε -net of the set $\{y_n\} = \{A\varphi_n\}$. But this contradicts the noncompactness of the sequence $\{y_n\}$. Suppose, for example, that the first of inequalities (6) is satisfied; then

$$\begin{aligned} 1 \geq (A_1\varphi_n, \varphi_n) &= \sum_1^{\infty} \lambda_i(A_1)|(\varphi_n, e_i)|^2 \geq \sum_{\nu}^{\infty} \lambda_i(A_1)|(\varphi_n, e_i)|^2 \geq \\ &\geq \frac{1}{\lambda_{\nu}(A_1)} \sum_{\nu}^{\infty} \lambda_i^2(A_1)|(\varphi_n, e_i)|^2 \geq \frac{1}{\lambda_{\nu}(A_1)} \varepsilon_0. \end{aligned}$$

The last inequality contradicts the fact that $\lambda_{\nu}(A_1) \rightarrow 0$ as $\nu \rightarrow \infty$. The theorem is proved.

Remark. Under the conditions of Theorem 2 the inequalities

$$\frac{1}{\sqrt{2}}((L_1 + L_2)y, y) \leq |(L_0 y, y)| \leq ((L_1 + L_2)y, y), \quad (9)$$

hold when $y \in D_0$; therefore the set E_0 is compact if and only if the set $E_1 = [y : ((L_1 + L_2)y, y) \leq 1, y \in D_0]$ is compact. Hence it is easy to obtain that the following assertion is valid: if the conditions of Theorem 2 are satisfied and the closure of the operator $L_1 + L_2$, the operator $\overline{L_1 + L_2}$, has finite defect indices, then the spectrum of the operator L_0 is discrete if and only if the operator $\overline{L_1 + L_2}$ has a self-adjoint extension with discrete spectrum.

2. Let I be some conjugation operator in the Hilbert space H . A linear operator A with domain of definition $D(A)$ dense in H will be called I -symmetric if, for any φ and ψ in $D(A)$, the equality $(A\varphi, I\psi) = (\varphi, IA\psi)$ holds, equivalent to the relation $A \subset IA^*I$. From the latter relation it follows that an I -symmetric operator admits a closure. If $A = IA^*I$, then the operator A is called I -self-adjoint ⁽¹⁾. In ⁽⁵⁾ it is shown that an I -symmetric operator admits an I -self-adjoint extension.

Combining the methods of P. S. Ismagilov ⁽³⁾ and Levinson ⁽²⁾, it is easy to prove that the minimal operator L_0 , generated by the differential expression

$$l[u] = -\frac{\partial}{\partial x} \left(a \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(b \frac{\partial u}{\partial y} \right) + gu \quad (10)$$

in $L_2(E_2)$, where a, b, g are complex-valued locally integrable functions of the variables x and y , $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, is I -self-adjoint if

$$\iint_{E_2} \frac{dx dy}{\sqrt{|a|M}} = +\infty, \quad \iint_{E_2} \frac{dx dy}{\sqrt{|b|M}} = +\infty; \quad (11)$$

here $M(r)$ ($r = \sqrt{x^2 + y^2}$) is a positive nondecreasing function on $(0, \infty)$;

$$\frac{\sqrt{|a|} \partial M / \partial x}{\sqrt{M^3}} < +\infty, \quad \frac{\sqrt{|b|} \partial M / \partial y}{\sqrt{M^3}} < +\infty, \quad \operatorname{Re} g + \operatorname{Im} g \geq -kM(r). \quad (12)$$

In particular, if the real and imaginary parts of a and b are positive, $\operatorname{Re} g \geq -c$, $\operatorname{Im} g \geq c$, and $\alpha \leq |a| \leq M$, $\alpha \leq |b| \leq M$, then the operator L_0 is I -self-adjoint and therefore has no residual spectrum ⁽¹⁾. Now, using Theorem 2, it is easy to prove that the operator L_0 has a discrete spectrum if and only if the minimal operator M_0 , generated by the differential expression

$$m[u] = -(\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2) + (\operatorname{Re} g + \operatorname{Im} g)u,$$

has a discrete spectrum. The operator M_0 has a discrete spectrum if and only if Molchanov's condition ⁽⁴⁾ is fulfilled. In the one-dimensional case it has the form

$$\int_x^{x+a} (\operatorname{Re} g + \operatorname{Im} g) dt \rightarrow \infty$$

as $x \rightarrow \infty$ and for every $a > 0$.

Using Theorem 1, it is easy to prove that the following holds:

Theorem 3. *Every I-self-adjoint extension of the minimal differential operator L_0 , generated by the differential expression*

by the expression

$$l[y] = -[(p_1(x) + ip_2(x))y']' + (g_1(x) + ig_2(x))y$$

in $L_2(0, \infty)$, has a discrete spectrum if at least one of conditions I-VI is satisfied:

I.

- a) $p_1(x) \geq \alpha > 0$;
- b) $g_1(x) \rightarrow +\infty$ as $x \rightarrow \infty$.

II.

- a) $p_1(x) \geq \alpha > 0$, $g_1(x) \geq -c$;
- b)

$$\int_x^{x+a} g_1(t) dt \rightarrow +\infty \quad \text{as } x \rightarrow \infty \text{ and for any } a > 0.$$

III.

- a) $p_1(x) > 0$, $g_1(x) \geq -c$;
- b)

$$\lim_{x \rightarrow \infty} x \int_x^\infty \frac{dt}{p_1(t)} = 0.$$

IV.

- a) $p_1(x) > 0$, $\int_0^\infty \frac{dt}{\sqrt{p_1(t)}} = c < \infty$;
- b) $g_1(x) - \frac{1}{16} [p_1'(x)]^2 / p_1(x) \geq \alpha > 0$.

V.

- a) $p_1(x) > 0$, $\int_0^\infty \frac{dt}{\sqrt{p_1(t)}} = \infty$;

b) $g_1(x) - 1/16 [p_1'(x)]^2/p_1(x) \rightarrow +\infty$ as $x \rightarrow \infty$.

VI.

a) $p_1(x) > 0, \quad g_1(x) \geq -c;$

b) $g_1(x) + 1/4 [p''(x) - 1/4 [p'(x)]^2/p(x)] \rightarrow +\infty$ as $x \rightarrow \infty$.

The assertion of Theorem 3 holds if conditions I-VI are imposed either on p_2, g_2 , or on $p_1 + p_2, g_1 + g_2$, or on $p_1 - p_2, g_1 - g_2$.

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