

# ON HEREDITARY AND COLLECTIVE NORMALITY OF A PERIPHERALLY BICOMPACT TREE-LIKE SPACE

V. V. Proizvolov

1970

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.71726>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 513.83

**MATHEMATICS**

**V. V. PROIZVOLOV**

## ON HEREDITARY AND COLLECTIVE NORMALITY OF A PERIPHERALLY BICOMPACT TREE-LIKE SPACE

*(Presented by Academician P. S. Aleksandrov, 28 I 1970)*

As numerous and simple examples show, a peripherally bicomact Hausdorff space need not be normal. Such a space, as Freudenthal showed ( $\wedge^1$ ), is only automatically completely regular. However, a tree-like\* peripherally bicomact space is normal, as is proved here. Moreover, the following theorems are valid.

**Theorem 1.** *Any two disjoint closed sets of a tree-like peripherally bicomact space are separated by a discrete set.*

**Proof.** In other words, it is necessary to prove that if  $A, B \subset X$  are closed, where  $X$  is peripherally bicomact and tree-like,  $A \cap B = \Lambda$ , then there exists a discrete set  $C \subset X$  such that  $X \setminus C = X_1 \cup X_2$ ,  $A \subset X_1$ ,  $B \subset X_2$ , where  $X_1$  and  $X_2$  are open in  $X$ ,  $X_1 \cap X_2 = \Lambda$ .

Any two points in a tree-like peripherally bicomact space are joined by a unique ordered continuum ( $\wedge^2$ ). If  $D \subset V \subseteq X$ , then by the linear hull of the set  $D$  in the set  $V$  we shall mean the union of all those ordered continua which join points of  $D$  pairwise and are entirely contained in  $V$ .

**Lemma 1.** *Let  $D \subset V \subseteq X$ , where  $D$  is closed,  $V$  is open, and  $X$  is a tree-like peripherally bicomact space. Then the linear hull  $LD$  of the set  $D$  in the set  $V$  is closed in  $X$ .*

We prove the lemma. Suppose that  $x \in [LD] \setminus LD$ . Take a connected neighborhood  $Ox$  with finite boundary such that  $Ox \cap D = \Lambda$ . Since  $x \in [LD] \setminus LD$ , there exists an infinite number of ordered continua entirely belonging to  $LD \cap [Ox]$  and joining pairwise boundary points of  $Ox$ . But the boundary of  $Ox$  is finite, and among these continua two distinct ones will be found joining a pair of boundary points of  $Ox$ , which contradicts Lemma 1 of ( $\wedge^2$ ).

**Lemma 2.** *The boundary of a connected open subset in a tree-like peripherally bicomact space is punctiform.*

**Lemma 3.** *If a point  $c$  separates a connected subset  $D$  of a tree-like peripherally bicomact space  $X$  between points  $a$  and  $b$ ,  $a, b \in D$ , then the point  $c$  also*

separates any subset containing  $D$  between the same points.

Returning to the proof of Theorem 1, take the linear hull  $LA$  of the set  $A$  in the set  $X \setminus B$  and then the linear hull  $LB$  of the set  $B$  in  $X \setminus LA$ . We decompose the locally connected set  $\tilde{X} = X \setminus (LA \cup LB)$  into open components:  $\tilde{X} = \bigcup_{\alpha} \tilde{X}_{\alpha}$ . It is further asserted that every  $[\tilde{X}_{\alpha}]$  contains not more than one point of  $LA$  and  $LB$ .

Indeed, if, for example,  $a_1, a_2 \in [\tilde{X}] \cap LA$ , then two possibilities arise. The first: the ordered continuum  $L$ , joining the points

---

\* A space is tree-like if it is connected and if any two points in it are separated by a third. It is automatically Hausdorff.

$a_1$  and  $a_2$ , and lying entirely in  $[\tilde{X}_{\alpha}]$ , such that  $L \cap LB = \Lambda$ . But then there is a contradiction with Lemma 2. Second: suppose there is a point  $b \in L \cap LB$ . But then, by Lemma 3, the point  $b$  separates  $[\tilde{X}_{\alpha}]$  between  $a_1$  and  $a_2$ , and also, as is easy to conclude,  $b$  separates  $\tilde{X}_{\alpha}$ , which contradicts the connectedness of  $\tilde{X}_{\alpha}$ .

Next, if  $[\tilde{X}_{\alpha}] \setminus \tilde{X}_{\alpha}$  consists of two points, then we take a point  $c_{\alpha} \in \tilde{X}_{\alpha}$  separating  $[\tilde{X}_{\alpha}]$  between these two points. The union of all such points  $c_{\alpha}$  forms a discrete set  $C$  closed in  $X$ .

The locally connected  $X \setminus C$  is decomposed into open components; those of them which intersect  $LA$  are united—we obtain  $OA$ ; those which intersect  $LB$  are united—we obtain  $OB$ . There is no component which would intersect both  $LA$  and  $LB$ . Suppose the contrary: suppose there is a component  $P$  of the set  $X \setminus C$  such that  $P \cap LA \neq \Lambda$  and  $P \cap LB \neq \Lambda$ . But then there is a component  $Q$  of the set  $P \setminus (LA \cup LB)$  such that  $[Q]$  intersects both  $LA$  and  $LB$ , in view of the connectedness of  $P$ . There is a component  $\tilde{X}_{\alpha}$  of the set  $X \setminus (LA \cup LB)$  containing  $Q$ . From the latter it follows that in the set  $C$  there is a point  $c_{\alpha}$  separating  $[Q]$  between the points

$$a = [Q] \cap LA$$

and

$$b = [Q] \cap LB,$$

and moreover  $c_{\alpha} \in Q$ , but this gives a contradiction. Thus the theorem is proved.

**Corollary 1.** *The large inductive dimension of a dendroidal peripherally bi-compact space is equal to one.\**

In addition, Theorem 1 implies the normality of a predendroidal peripherally bicomact space. But a stronger assertion is true.

**Theorem 2.** *A dendroidal peripherally bicomact space is hereditarily normal.*

**Proof.** Let  $\tilde{X} \subseteq X$  be a subspace of the dendroidal peripherally bicomact space  $X$ , and let  $A, B \subset \tilde{X}$  be closed in  $\tilde{X}$  and  $A \cap B = \Lambda$ . Consider the set  $C = [A] \cap [B]$ . Obviously,  $C \subseteq X \setminus \tilde{X}$ . We decompose the locally connected open set  $X \setminus C$  in  $X$  into open components:

$$X \setminus C = \bigcup_{\alpha} X_{\alpha}.$$

Each  $X_{\alpha}$  is a peripherally bicomact dendroidal subspace of  $X$ . Indeed, every subset of a peripherally bicomact dendroidal space is itself peripherally bicomact, since in a peripherally bicomact dendroidal space every point has arbitrarily small neighborhoods with finite boundaries <sup>(3)</sup>. Therefore, by Theorem 1, for the sets  $A_{\alpha} = A' \cap X_{\alpha}$  and  $B_{\alpha} = B' \cap X_{\alpha}$ , where  $A' = [A] \setminus C$  and  $B' = [B] \setminus C$ , there exist disjoint neighborhoods  $OA_{\alpha}$  and  $OB_{\alpha}$  such that  $OA_{\alpha}, OB_{\alpha} \subset X_{\alpha}$ . Let us also note that  $A \subseteq A'$  and  $B \subseteq B'$ .

It is not difficult to see that

$$OA = \bigcup_{\alpha} OA_{\alpha}$$

and

$$OB = \bigcup_{\alpha} OB_{\alpha}$$

are disjoint neighborhoods of the sets  $A$  and  $B$ . This means that  $\tilde{X}$  is normal, and Theorem 2 is proved.

**Corollary 2.** *A dyadic dendroidal bicomactum is metrizable.*

We had put forward a hypothesis on the metrizability of a hereditarily normal dyadic bicomactum, which B. A. Efimov succeeded in proving <sup>(4)</sup>. In combination with Theorem 2 this gives Corollary 2.

**Theorem 3.** *A dendroidal peripherally bicomact space is collectively normal.*

**Lemma 4.** *The set  $LD$  mentioned in Lemma 1 is locally connected.*

Let us prove the lemma. Let  $x \in LD$  be a point and let  $Ox$  be its connected neighborhood in  $X$  such that  $Ox \subset V$ . It is asserted that the set  $O_1x = Ox \cap LD$ , which is a neighborhood of the point  $x$  in the set  $LD$ , is connected. Indeed, every point  $y \in O_1x$  can be joined to the point  $x$  by an ordered continuum lying entirely in  $Ox$ , and hence also lying in  $O_1x$  in the sense of the linear hull. The lemma is proved.

\* The question of Ind of a dendroidal peripherally bicomact space was raised by B. A. Pasynkov.

**Lemma 5.** Let  $\Gamma \subset X$  be a discrete subset of the tree-like peripherally bicomact space  $X$ , and suppose that  $X \setminus \Gamma$  is connected. Then there exists a closed set  $C \subset X$  such that  $X \setminus C$  is decomposed into open components, each of which contains at most one point of  $\Gamma$ .

We prove the lemma. Let  $\Gamma = \{x_\alpha\}$ ; for an arbitrary  $x_\alpha \in \Gamma$  denote  $\Gamma_\alpha = \Gamma \setminus x_\alpha$ . The linear hull  $L\Gamma_\alpha$  of the set  $\Gamma_\alpha$  in  $X$  is closed by Lemma 1, is connected, and  $x_\alpha \notin L\Gamma_\alpha$ . The latter follows from the fact that  $X \setminus x_\alpha$  is connected by the hypothesis of the lemma, and any two points in  $X \setminus x_\alpha$  are joined by a unique ordered continuum (see Lemma 1 from <sup>(2)</sup>). By  $V_\alpha$  we denote the open component of the set  $X \setminus L\Gamma_\alpha$  containing the point  $x_\alpha$ . It is asserted that  $V_{\alpha_1} \cap V_{\alpha_2} = \Lambda$ . Suppose the contrary: there is a point  $c \in V_{\alpha_1} \cap V_{\alpha_2}$ . Then  $V_{\alpha_1} \cup V_{\alpha_2}$  is connected. We note that  $[V_{\alpha_1}] \setminus V_{\alpha_1}$  consists of a single point; denote it by  $y_{\alpha_1}$ . This latter circumstance can be justified with the help of Lemma 3. Correspondingly, by  $y_{\alpha_2}$  denote the point  $[V_{\alpha_2}] \setminus V_{\alpha_2}$ . From what has been said it is clear that

$$[V_{\alpha_1} \cup V_{\alpha_2}] \setminus (V_{\alpha_1} \cup V_{\alpha_2}) = y_{\alpha_1} \cup y_{\alpha_2}$$

and  $V = [V_{\alpha_1} \cup V_{\alpha_2}]$  is a tree-like peripherally bicomact subspace of the space  $X$ . We note that  $y_{\alpha_2} \notin V_{\alpha_1}$ . Indeed, if  $y_{\alpha_2} \in V_{\alpha_1}$ , then we would join  $x_{\alpha_2}$  with  $y_{\alpha_1}$  by an ordered continuum, which necessarily would pass through  $y_{\alpha_2}$ , and then join the point  $y_{\alpha_1}$  with  $x_\beta \in \Gamma \setminus (x_{\alpha_1}, x_{\alpha_2})$ ; consequently, we would obtain that  $x_{\alpha_2}$  and  $x_\beta$  are joined by an ordered continuum to which the point  $y_{\alpha_2}$  belongs, which means:  $y_{\alpha_2} \in L\Gamma_{\alpha_1}$ . But the latter at the same time means that  $y_{\alpha_2} \notin V_{\alpha_1}$ . Accordingly,  $y_{\alpha_1} \notin V_{\alpha_2}$ . However, the point  $y_{\alpha_1} \in L\Gamma_{\alpha_1}$  and the point  $y_{\alpha_2} \in L\Gamma_{\alpha_1}$ , for, joining  $x_{\alpha_2}$  and  $x_\beta$  by an ordered continuum, we see that it passes through  $y_{\alpha_2}$ . The ordered continuum  $L_1$ , joining  $x_\beta$  with  $y_{\alpha_1}$ , does not intersect  $V_{\alpha_1}$ , since  $x_\beta, y_{\alpha_1} \in L\Gamma_{\alpha_1}$ , and, moreover,  $L_1 \cap V_{\alpha_2} = \Lambda$ , since the continuum joining  $x_\beta$  with  $x_{\alpha_1}$  belongs entirely to  $L\Gamma_{\alpha_2}$  and contains  $L_1$  in itself. Thus,

$$L_1 \cap (V_{\alpha_1} \cup V_{\alpha_2}) = \Lambda.$$

Similarly,

$$L_2 \cap (V_{\alpha_1} \cup V_{\alpha_2}) = \Lambda,$$

where  $L_2$  is the ordered continuum joining the points  $x_\beta$  and  $y_{\alpha_2}$ . Denote by  $L$  the connected closed set  $L_1 \cup L_2$ . It is seen that the connected closed sets  $L$  and  $[V]$  intersect in two points  $y_{\alpha_1}$  and  $y_{\alpha_2}$ , which contradicts the fact that in a tree-like peripherally bicomact space the intersection of connected closed sets is empty or connected (see Lemma 2 from <sup>(2)</sup>). Hence,  $V_{\alpha_1} \cap V_{\alpha_2} = \Lambda$ , and as the set  $C$  we take  $X \setminus \bigcup V_\alpha$ . Lemma 5 is proved.

**Proof of Theorem 3.** Let  $\{A_\alpha\}$  be a discrete family of closed subsets of the tree-like peripherally bicomact space  $X$ . By transfinite induction we extend each  $A_\alpha$  to some closed locally connected  $\tilde{A}_\alpha$  in the following way. The set

$$\tilde{A}_1 = LA_1$$

in the set

$$X \setminus \bigcup_{\alpha \neq 1} A_\alpha.$$

If all  $\tilde{A}_\alpha$  have been constructed for all  $\alpha < \beta$ , then  $\tilde{A}_\beta$  is defined as follows:  $\tilde{A}_\beta$  is the union of all those ordered continua joining pairs of points of  $A_\beta$  which do not intersect the closed set

$$\bigcup_{\alpha \leq \beta} \tilde{A}_\alpha \cup \bigcup_{\alpha \neq \beta} A_\alpha.$$

In order that the inclusion  $A_\alpha \subset \tilde{A}_\alpha$  hold, by definition we adjoin to  $\tilde{A}_\alpha$  all points of  $A_\alpha$  itself.

By Lemmas 1 and 4, each  $\tilde{A}_\alpha$  is a locally connected closed set in  $X$ . Decompose  $\tilde{A}_\alpha$  into components; we obtain a discrete system of closed sets:

$$\tilde{A}_\alpha = \bigcup_{\gamma} \tilde{A}_{\alpha\gamma}.$$

The system  $\{\tilde{A}_{\alpha\gamma}\}$ , over all admissible  $\alpha$  and  $\gamma$ , is a discrete system of connected closed sets.

The subspace

$$\tilde{X} = X \setminus \bigcup_{\alpha, \gamma} \tilde{A}_{\alpha\gamma}$$

is decomposed into open components:

$$\tilde{X} = \bigcup_{\beta} \tilde{X}_\beta.$$

At the same time,  $[\tilde{X}_\beta]$  for any  $\beta$  contains at most one point from each  $\tilde{A}_{\alpha\gamma}$ ; this means that

$$\Gamma_\beta = [\tilde{X}_\beta] \setminus \tilde{X}_\beta$$

is a discrete set. Applying Lemma 5 to the subspace  $[\tilde{X}_\beta]$  and its discrete subset  $\Gamma_\beta$ , we obtain a closed set  $C_\beta$ ,  $C_\beta \subset \tilde{X}_\beta$ . It is possible

to show that the set  $C = \bigcup_{\beta} C_{\beta}$  is closed. Suppose the contrary: there is a point  $x \in [C] \setminus C$ . Take an arbitrary neighborhood  $O_x$  of it with finite boundary. The neighborhood  $O_x$  intersects infinitely many sets of the form  $\widetilde{X}_{\beta}$ , but since the boundary of  $O_x$  is finite and the family  $\{\widetilde{X}_{\beta}\}$  is disjoint, this means that infinitely many sets of the form  $\widetilde{X}_{\beta}$  are entirely contained in  $O_x$ . Thus every neighborhood of the point  $x$  with finite boundary contains infinitely many sets of the form  $\widetilde{X}_{\beta}$  and, consequently, intersects infinitely many sets of the form  $\widetilde{A}_{\alpha}$ , which contradicts the discreteness of the family  $\{\widetilde{A}_{\alpha}\}$ . Hence  $C$  is closed. After this it is not hard to see that  $X \setminus C$  decomposes into open components, each of which contains at most one set of the form  $\widetilde{A}_{\alpha}$ ; from this it follows at once that the discrete family  $\{A_{\alpha}\}$  is placed in a system of pairwise disjoint neighborhoods. Consequently,  $X$  is collectively normal. The theorem is proved.

**Example.** There exist dendroid spaces that are not normal, as the following example shows\*. On the half-plane we introduce a new topology as follows. If  $x \in L$ , where  $L$  is the boundary line, then a basic neighborhood  $O_x$  is an open disk tangent to  $L$  at the point  $x$ , together with the point  $x$ . If, however,  $x \notin L$ , then its basic neighborhood is an arbitrary interval containing  $x$  and perpendicular to  $L$ . The space obtained in this way is dendroid, completely regular, but not normal (compare it with the Niemytzki space!).

Faculty of Mechanics and Mathematics  
M. V. Lomonosov Moscow State University

Received  
26 XII 1970

## References

1. H. Freudenthal, *Ind. Math. Amst.*, **13**, 184 (1951).
2. V. V. Prizvolov, *DAN*, **189**, No. 4 (1969).
3. G. L. Gurin, *Vestn. MGU*, No. 1, 9 (1969).
4. B. A. Efimov, *Tr. Mosk. matem. obshch.*, **14**, 211 (1965).

\* The example answers a question raised by O. V. Lokutsievskii.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*