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Abstract

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MATHEMATICS

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ON THE BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS IN WEIGHTED SPACES

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1°. Let R^n be n -dimensional Euclidean space of points $x = (x_1, \dots$

$$\dots, x_n); \quad \rho^2 = |x|^2 = \sum_{i=1}^n x_i^2; \quad S = \{x : |x| = 1\}; \quad \theta = x\rho^{-1} \in S.$$

If $\omega = (\omega_1, \dots, \omega_n)$ is a set of nonnegative integers (a multi-index), then

$$D_x^\omega = D_{x_1}^{\omega_1} \dots D_{x_n}^{\omega_n}; \quad D_{x_i} = \partial/\partial x_i; \quad x^\omega = x_1^{\omega_1} \dots x_n^{\omega_n};$$

$$|\omega| = \sum_{i=1}^n \omega_i; \quad \omega! = \omega_1! \dots \omega_n!$$

Denote by D the space of infinitely differentiable functions with compact supports. We define the space $L_{p,\alpha}$ as the set of functions defined on R^n with finite norm

$$\|u\|_{L_{p,\alpha}} = \|\rho^\alpha u\|_{L_p} < \infty, \quad 1 < p < \infty.$$

We shall specify a singular integral operator with symbol $\Phi(x, \xi)$ in the form

$$Au = F_{\xi \rightarrow x}^{-1} \Phi(x, \xi) F_{y \rightarrow \xi} u, \quad (1)$$

where $F_{y \rightarrow \xi} u$ is the Fourier transform of the function $u \in D$. We shall also use the representation of a singular operator by means of a kernel, namely:

$$Au = \int_{R^n} K(x, x-y) u(y) dy, \quad (1')$$

where $K(x, x-y) = f(x, \theta_{xy})|x-y|^{-n}$, $\theta_{xy} = (x-y)|x-y|^{-1}$.

The boundedness of the one-dimensional singular integral in the space $L_{p,\alpha}$ ($-1/p < \alpha < 1/p'$; $1/p + 1/p' = 1$) was proved by K. I. Babenko ⁽²⁾. E. M. Stein ⁽³⁾ obtained a theorem on the boundedness of the operator (1') in the space $L_{p,\alpha}$ for arbitrary n and $\alpha \in (-n/p, n/p')$ under the condition that $\text{vraisup} |f(x, \theta)| < \infty$. B. A. Plamenevskii ⁽⁴⁾ and the author ⁽⁵⁾ independently proved that, for $\alpha \in (-n/2, n/2)$, $|\alpha| \neq n/2 + k$ ($k = 0, 1, \dots$), the operator (1) with symbol $\Phi(\xi)$, defined on certain dense subsets in $L_{2,\alpha}$, is bounded in the space $L_{2,\alpha}$.

The present paper is devoted to generalizing these results in various directions.

Denote by Q_s the set of functions $u \in D$ satisfying the conditions

$$\int_{R^n} x^\omega u(x) dx = 0,$$

where ω is any multi-index of order $0 \leq |\omega| \leq s$. By Q_{-s} we denote the set of functions $u \in D$ such that $0 \notin \text{supp } u$, and satisfying

conditions

$$\int_0^\infty \rho^{-q} u(\rho, \theta) d\rho = 0, \quad q = 1, 2, \dots,$$

The sets Q_s and Q_{-s} are dense in the space $L_{p,\alpha}$.

Theorem 1.1. I. Let $n/p < \alpha < n/p'$ and

$$\int_S |f(x, \theta)|^{p'} d\theta \leq \text{const} < \infty. \quad (2)$$

Then the operator (1') is bounded in the space $L_{p,\alpha}$.

II. Let $\alpha > n/p'$, $\alpha \neq n/p' + k$ ($k = 1, 2, \dots$), condition (2) be satisfied, and

$$\int_{R \leq |x| \leq 2R} |D_z^\omega K(x, z)|_{z=x}^p dx \leq \text{const} \cdot R^{-n(p-1)-(\omega)p},$$

where ω is any multi-index of order $0 \leq |\omega| \leq s = [\alpha - n/p']^*$ and $R \in (0, \infty)$. Suppose further that for some $\varkappa \in (\alpha - n/p', s + 1]$ the inequality

$$\int_{R \leq |x| \leq 2R} |K^+(x, y)|^p dx \leq \text{const} \cdot R^{-n(p-1)} \left(\frac{|y|}{R} \right)^{\varkappa p},$$

is satisfied, where

$$K^+(x, y) = K(x, x - y) - \sum_{|\omega|=0}^s D_z^\omega K(x, z)|_{z=x} \frac{y^\omega}{\omega!}.$$

Then the operator $(1')$, defined on the set $Q(s)$, is bounded in the space $L_{p,\alpha}$.

III. Let $\alpha < -n/p$, $|\alpha| \neq n/p + k$ ($k = 1, 2, \dots$), condition (2) be satisfied, and

$$\int_S |D_x^\omega K(x, x - y)|_{x=0}|^{p'} d\theta \leq \text{const} \cdot |y|^{-(n+|\omega|)p'},$$

where ω is a multi-index of order $0 \leq |\omega| \leq s = [|\alpha| - n/p]$. Suppose further that for $|y| > 2|x|$ and $\varkappa \in (|\alpha| - n/p, s + 1]$ the inequality

$$\int_S |K^-(x, y)|^{p'} d\theta \leq \text{const} \cdot |y|^{-np'} \left(\frac{|x|}{|y|} \right)^{\varkappa p'},$$

is satisfied, where

$$K^-(x, y) = K(x, x - y) - \sum_{|\omega|=0}^s D_x^\omega K(x, x - y)|_{x=0} \frac{x^\omega}{\omega!}.$$

Then the operator $(1')$, defined on the set Q_{-s} , is bounded in the space $L_{p,\alpha}$.

For $\alpha > n/p'$ the operator $(1')$, defined on the set Q_s , $s = [\alpha - n/p']$, coincides with the operator

$$A^+ u = \int_{R^n} K^+(x, y) u(y) dy.$$

For $\alpha < -n/p$ the operator $(1')$, defined on the set Q_{-s} , $s = [|\alpha| - n/p]$, coincides with the operator

$$A^- u = \int_{R^n} K^-(x, y) u(y) dy.$$

2°. Let us consider separately the case of the space $L_{2,\alpha}$. By definition, $\Phi(x, \xi) \in L_\infty W_2^l(S)$ if $\Phi(x, \xi)$ belongs to the space $W_2^l(S)$

* Here and below $[r]$ denotes the integer part of the number r .

(the space of S. L. Sobolev—L. N. Slobodetskii on the sphere S) with respect to the variable ξ and

$$\text{vrai sup}_{x \in R^n} \|\Phi(x, \xi)\|_{W_2^l(s)} < \infty.$$

Theorem 2. Let $\Phi(x, \xi) \in L_\infty W_2^l(S)$, $l > (n-1)/2 + |\alpha|$, $\alpha \neq n/2 + k$ ($k = 0, 1, \dots$). Then the operator (1), defined on the whole space $L_{2,\alpha}$ for $|\alpha| < n/2$ and on the set $Q_s(Q_{-s})$ for $|\alpha| > n/2$ ($s = \lfloor |\alpha| - n/2 \rfloor$), is bounded in the space $L_{2,\alpha}$.

The proof is based on the following estimate

$$\sum_{m=0}^{m_k} \|F_{\xi \rightarrow x}^{-1} Y_{mk}(\xi) F_{y \rightarrow \xi} u\|_{L_{2,\alpha}}^2 \leq \text{const} \cdot k^{n-2+2|\alpha|} \|u\|_{L_{2,\alpha}}^2,$$

where $Y_{mk}(\xi)$ is a spherical function of order k .

For $\alpha = 0$, theorem 2 becomes the theorem of S. G. Mikhlin ⁽¹⁾—M. S. Agranovich ⁽⁷⁾. There is an assertion analogous to theorem 2 in the spaces L_2 with weight function $\lambda(x) = (1 + |x|)^\alpha$ and with weight function $\lambda(x)$ equal to $|x|^\alpha$ for $|x| \leq 1$ and to unity for $|x| > 1$.

3°. Let H^μ be the function space defined in ⁽⁶⁾, endowed with the norm

$$\|u\|_{H^\mu} = \left(\int_{R^n} \mu^2(\xi) |F_{x \rightarrow \xi} u|^2 d\xi \right)^{1/2} < \infty.$$

We shall say that $a(x, \xi) \in H^\mu L_\infty$ if $a(x, \xi)$ belongs to the space H^μ with respect to the variable x and

$$\text{vrai sup}_{\xi \in R^n} \|a(\cdot, \xi)\|_{H^\mu} < \infty.$$

The class of functions $L_\infty H^\mu$ is defined analogously.

We define the space $L_2(\mu)$ as the set of functions given on R^n for which the norm

$$\|u\|_{L_2(\mu)} = \left(\int_{R^n} |\mu(x) u(x)|^2 dx \right)^{1/2} < \infty$$

is finite.

Theorem 3. The pseudodifferential operator

$$Bu = F_{\xi \rightarrow x}^{-1} b(x, \xi) F_{y \rightarrow \xi} u \quad (3)$$

is bounded in the space L_2 if $b(x, \xi) \in H^\mu L_\infty$ and

$$\int_{R^n} \mu^{-2}(\tau) d\tau < \infty. \quad (4)$$

Condition (4) is, in a certain sense, sharp. Namely, for any function μ not satisfying condition (4), one can find a symbol $b(x) \in H^\mu$ such that the operator (3) is not bounded in L_2 . A close result for singular operators, refining the theorem of S. G. Mikhlin–M. S. Agranovich mentioned above, was obtained in ⁽¹¹⁾.

Theorem 4. Let a pseudodifferential operator with symbol $b(x, \xi)$ be bounded in the space $L_2(\nu)$, and let the function $\varphi(x, \xi)$ belong to $L_\infty H^\mu$.

Then the pseudodifferential operator with symbol $\varphi(x, \xi)b(x, \xi)$ is bounded in the space $L_2(\nu)$, if

$$\int_{R^n} \nu^2(x+y)\mu^{-2}(y) dy \leq \text{const} \cdot \nu^2(x).$$

The fact that the continuity conditions for pseudodifferential operators in L_2 can at the same time be multiplier conditions was first noted in the works of M. Sh. Birman and M. Z. Solomyak ^(9,10).

Theorem 3 refines, and Theorem 4 generalizes, analogous assertions from works ^(9, 10), obtained by methods of the theory of double operator integrals.

4°. By $L_{p,\alpha,\beta,\gamma}$ we denote the completion of the space of functions $u \in D$ ($0 \notin \text{supp } u$) with respect to the norm

$$\|u\|_{L_{p,\alpha,\beta,\gamma}} = \|\rho^\alpha(-\Delta)^{\beta/2}(-\delta)^{\gamma/2}u\|_{L_p},$$

where Δ is the Laplace operator, δ is the spherical part of Δ , and α, β, γ are real numbers.

Let us formulate some properties of the spaces $L_{p,\alpha,\beta,\gamma}$.

1) Let $-n/p < \alpha - \beta < \alpha < n/p'$. Then, for functions $u \in D$, the inequalities

$$c_1 \|\rho^\alpha(-\Delta)^{\beta/2}u\|_{L_p} \leq \|(-\Delta)^{\beta/2}\rho^\alpha u\|_{L_p} \leq c_2 \|\rho^\alpha(-\Delta)^{\beta/2}u\|_{L_p},$$

hold, where c_1, c_2 are constants independent of the function u .

2) Let $0 < \beta \neq n/p + k$, $k = 0, 1, \dots$. Then for all functions $u \in D$ ($0 \notin \text{supp } u$) the inequalities

$$c_1 \|(-\Delta)^{\beta/2}u\|_{L_p} \leq \|(-\Delta_\rho)^{\beta/2}u\|_{L_p} + \|\rho^{-\beta}(-\delta)^{\beta/2}u\|_{L_p} \leq c_2 \|(-\Delta)^{\beta/2}u\|_{L_p}, \quad (5)$$

are valid, where Δ_ρ denotes the radial part of the Laplace operator.

In the proof of inequality (5) the estimate

$$\|\rho^{-\beta}u\|_{L_p} \leq c\|(-\Delta)^{\beta/2}u\|_{L_p},$$

is used, where $0 < \beta \neq n/p + k$, $k = 0, 1, \dots$, previously proved by V. P. Il' in (8) for $0 < \beta < n/p$.

With the aid of the theorem from 1° and the formulated properties of the spaces $L_{p,\alpha,\beta,\gamma}$, one proves theorems on the boundedness of operators whose symbols, for large $|\xi|$, have the form

$$\sigma(x, \xi) = \sum_{k=1}^N |\xi|^{\lambda_k} \sigma_k(x, \xi),$$

where λ_k are real numbers, and $\sigma_k(x, \xi)$ are functions positively homogeneous of degree zero, sufficiently smooth and, for large $|x|$, independent of x .

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