

ON ISOMETRIC OPERATORS ON SUBSPACES OF (L^p)

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Abstract

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MATHEMATICS

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ON ISOMETRIC OPERATORS ON SUB-SPACES OF L^p

(Presented by Academician V. I. Smirnov on 28 XI 1969)

In this note we generalize and strengthen the results obtained in paper (3), in which some information on the history of the question is also given.

1. Let (X_1, σ_1) and (X_2, σ_2) be two spaces with positive normalized measures and $0 < p < \infty$. We shall consider complex spaces $L^p(\sigma_j)$, $j = 1, 2$; for brevity of formulation, the number

$$\left(\int_{X_j} |f|^p d\sigma_j \right)^{1/p}$$

will be called the L^p -norm of the function $f \in L^p(\sigma_j)$ also in the case $0 < p < 1$.

The basis for all that follows is the following

Lemma 1. *Let p be not an even integer and let $f_j \in L^p(\sigma_j)$, $j = 1, 2$. Suppose that for every (complex) z*

$$\int_{X_1} |1 + zf_1|^p d\sigma_1 = \int_{X_2} |1 + zf_2|^p d\sigma_2. \quad (1)$$

Then $|f_1|$ and $|f_2|$ are equidistributed, i.e. for all $\lambda \geq 0$

$$\sigma_1(\{|f_1| \geq \lambda\}) = \sigma_2(\{|f_2| \geq \lambda\}).$$

Proof. Let φ and ψ be bounded functions on $[0, 1]$. Put

$$h_1(z) = \int_0^1 \psi(t) dt \int_0^{2\pi} |1 + z\varphi(t)e^{i\theta}|^p d\theta,$$

$$h(z) = \int_1^2 ds \int_1^2 h_1(stz) dt.$$

Then h is twice continuously differentiable on $(0, \infty)$, $h(z) = h(|z|)$, and for all $r \geq 0$

$$\int_{X_1} h(r f_1) d\sigma_1 = \int_{X_2} h(r f_2) d\sigma_2. \quad (2)$$

Moreover, the functions φ and ψ can be chosen so that there exists a number q , $0 < q < p$, such that

$$\int_0^\infty |h(x)| x^{-q-1} dx < \infty.$$

Let H be the Mellin transform of h ; then H is summable on the line $\operatorname{Re} \zeta = -q$ and, for $x \geq 0$,

$$h(x) = \int_{\operatorname{Re} \zeta = -q} H(\zeta) x^{-\zeta} d\zeta.$$

Hence, for $j = 1, 2$ and $r \geq 0$ we have

$$\int_{X_j} h(r f_j) d\sigma_j = \int_{X_j} h(r |f_j|) d\sigma_j = \int_{\operatorname{Re} \zeta = -q} H(\zeta) G_j(\zeta) r^{-\zeta} d\zeta, \quad (3)$$

where

$$G_j(\zeta) = \int_{X_j} |f_j|^{-\zeta} d\zeta.$$

Taking (2) into account, from (3) we obtain that $G_1 = G_2$ on the line $\operatorname{Re} \zeta = -q$. Put $a_j(\lambda) = \sigma_j(\{|f_j| \geq \lambda\})$; then

$$G_j(\zeta) = (-\zeta) \int_0^\infty \lambda^{-\zeta-1} a_j(\lambda) d\lambda, \quad \operatorname{Re} \zeta = -q,$$

and, applying the inverse Mellin transform, we obtain $a_1 = a_2$, as was required.

Let now B be a subspace in $L^\infty(\sigma_1)$ containing the constants (not necessarily closed). From Lemma 1 it follows easily

Lemma 2. If p is not an even integer and a linear operator T maps B into $L^p(\sigma_2)$ preserving the L^p -norms, and moreover $T1 = 1$, then T preserves the L^s -norms for every $s > 0$ and the L^∞ -norms.

If p is an even integer, then the assertion of Lemma 1 is false. Therefore, in the case of an even integer p , generally speaking, the assertion of Lemma 2 is also false unless additional conditions are imposed on the subspace B . We shall say that B has property (α_m) , $m = 1, 2, \dots$, if there exist no more than m subalgebras with 1 in $L^\infty(\sigma_1)$ such that they are contained in B , while their (uniformly) closed linear span contains B .

Lemma 3. If p is an even integer, $p \neq 2$, then the assertion of Lemma 2 is valid under the condition that B has property $(\alpha_{p/2})$.

With the aid of Lemmas 2 and 3 (and also Lemma 2 from [3]) the following theorem on the extension of isometric operators defined on subspaces in L^p is proved.

Theorem 1. Let a linear operator T map B into $L^p(\sigma_2)$ with preservation of the L^p -norms, and let $p \neq 2$. If p is an even integer, we shall assume that B has property $(\alpha_{p/2})$. Let $F = T1$ and let $\widetilde{B}$ be the smallest symmetric subalgebra in $L^\infty(\sigma_1)$ containing B . Then there exists a linear operator $\widetilde{T}$ on $\widetilde{B}$ such that:

- 1) $\widetilde{T}$ maps $\widetilde{B}$ L^p -isometrically into $L^p(\sigma_2)$;
- 2) $\widetilde{T}$ has the form

$$\widetilde{T}f = F\varphi(f), \quad f \in \widetilde{B},$$

where φ is some L^∞ -isometric symmetric homomorphism of the algebra $\widetilde{B}$ into $L^\infty(\sigma_2)$;

- 3) the restriction of $\widetilde{T}$ to B coincides with T .

Remark. All the preceding assertions, with the obvious changes in their formulations, are also valid for real L^p ; we note only that, for even integer p , in this case it is enough to require the condition (α_p) .

2. Theorem 1 can be applied to find the general form of isometric transformations on certain subspaces in L^p . First of all we observe that from this theorem one easily obtains a proof of the well-known Banach-Lamperti theorem ^(1,2) on the general form of isometric transformations on the whole of L^p . Further, if the subspace B is such that the algebra $\widetilde{B}$ is dense in L^p , then the problem of describing L^p -isometric transformations of B is reduced, in view of Theorem 1, to finding such isometric operators on the whole of L^p with respect to which B is invariant.

Let D_1 and D_2 be bounded domains in C^n . Denote by $L_a^p(D_j)$, $j = 1, 2$, the subspace of $L^p(D_j, d\sigma_j)$ consisting of functions analytic in D_j . Here $d\sigma_j$ is the normalized Lebesgue measure in D_j .

Theorem 2. Suppose that $\text{int } \overline{D}_1 = D_1$ ($\text{int } \overline{D}_1$ is the interior of $\overline{D}_1$) and that $L_a^\infty(D_1)$ is dense in $L_a^p(D_1)$. Let T be an isometric embedding of the space $L_a^p(D_1)$ into $L_a^p(D_2)$. Then, if $p \neq 2$, there exists a holomorphic mapping $\tau : D_2 \rightarrow \overline{D}_1$ having the following properties:

- 1) τ biholomorphically maps the domain

$$D_2 \setminus \{\lambda : \lambda \in D_2, \tau'(\lambda) = 0\} \stackrel{\text{def}}{=} D_2 \setminus J,$$

where τ' is the Jacobian of τ , onto a domain in D_1 whose complement to D_1 has measure 0;

- 2) $(\tau')^{2/p}$ exists as a single-valued holomorphic function in D_2 ;
- 3) for every $f \in L_a^p(D_1)$

$$Tf = e^{i\gamma(V_2/V_1)^{1/p}(\tau')^{2/p}} f(\tau) \quad (4)$$

on $D_2 \setminus J$ and $Tf = 0$ on J , where γ is a real constant, and V_j is the volume of D_j , $j = 1, 2$.

Conversely, for any holomorphic mapping $\tau : D_2 \rightarrow \overline{D_1}$ having properties 1) and 2), and any real γ , formula (4) defines an isometric embedding $T : L_a^p(D_1) \rightarrow L_a^p(D_2)$.

Remark. For $n = 1$ and any $p \neq 2$, or for any n but irrational p , the formulation of Theorem 2 is simplified. Namely, in this case $J = \emptyset$ (and, consequently, τ maps D_2 into D_1). If $n > 1$ and p is rational, then, as simple examples show, J may be nonempty.

Theorem 2 easily implies

Theorem 3. Suppose that $\text{int } \overline{D_j} = D_j$ and $L_a^\infty(D_j)$ is dense in $L_a^p(D_j)$, $j = 1, 2$. If the spaces $L_a^p(D_1)$ and $L_a^p(D_2)$ are isometrically isomorphic, then the domains D_1 and D_2 are biholomorphically equivalent.

3. Let now D_1 and D_2 be bounded domains in R^n , $n > 1$. Denote by $L_h^p(D_j)$, $j = 1, 2$, the subspace in $L^p(D_j, d\sigma_j)$ consisting of functions harmonic in D_j .

Theorem 4. If $\text{int } \overline{D_1} = D_1$ and the space $L_h^p(D_1)$ is isometrically isomorphic to some subspace in $L_h^p(D_2)$, $p \neq 2$, then there exists a domain $G_1 \subset D_1$ such that the measure of $D_1 \setminus G_1$ is 0, and, for $p \neq n/(n-2)$, the domains G_1 and D_2 coincide up to a similarity transformation,* while, if $p = n/(n-2)$, these domains coincide up to similarity transformations and inversion.

If one assumes that $L_h^\infty(D_1)$ is dense in $L_h^p(D_1)$, then in the case when p is not an even integer one can show that every isometric embedding operator of $L_h^p(D_1)$ into $L_h^p(D_2)$ is generated by some similarity transformation (or by a similarity and inversion, for $p = n/(n-2)$). Hence follows

Corollary. Let D be a bounded domain in R^n such that $\text{int } \overline{D} = D$ and $L_h^\infty(D)$ is dense in $L_h^p(D)$. If p is not an even integer, then the space $L_h^p(D)$ has no proper subspace isometrically isomorphic to all of $L_h^p(D)$.

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CITED LITERATURE

¹ S. Banach, *Course of Functional Analysis*, Kyiv, 1948. ² J. Lamperti, Pacific J. Math., 8, No. 3, 459 (1958). ³ A. I. Plotkin, DAN, 185, No. 5 (1969).

* By a similarity transformation here is meant the product of a homothety and a congruent transformation.

Note: Figure translations are in progress. See original paper for figures.

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