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B. A. Pasynkov, I. K. Lifanov

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Abstract

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MATHEMATICS

B. A. PASYNKOV, I. K. LIFANOV

EXAMPLES OF BICOMPACTA WITH NON-COINCIDING INDUCTIVE DIMENSIONS

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In ⁽¹⁾ V. Filippov constructed a bicom pactum X with $\text{ind } X = 2 < \text{Ind } X = 3$, but the bicom pactum X has cardinality 2^{2^c} . We constructed a bicom pactum T^3 of this note and B. A. Pasynkov constructed bicom pacta T^2 and T_1^2 of the note ⁽²⁾ with noncoinciding inductive dimensions and of cardinality c . After becoming acquainted with the bicom pactum T^3 , applying the device described in item 4 of this note and used by us for the construction of the bicom pactum T^3 , V. Filippov constructed a bicom pactum with the first axiom of countability and with noncoinciding inductive dimensions. Soon after this, both V. Filippov and we constructed simpler bicom pacta with the first axiom of countability and with noncoinciding inductive dimensions (see ^(2, 3) and the bicom pacta S^3 of this note).

It is appropriate to note that the constructions of T^3 and S^3 coincide up to replacing the tail χ_1 by the tail χ_2 (see items 5 and 5').

1. By $R = \{r\}$ and $I = \{i\}$ we shall denote respectively the sets of rational and irrational points of the segment $Q^1 = [0, 1]$. Represent the set R as the disjoint sum of two everywhere dense in Q^1 sets: $R_0 = \{r_0\}$ and $R_1 = \{r_1\}$.
2. Consider the cube $Q^3 = Q^1 \times Q^1 \times Q^1$. For a point $x = (t_0^1, t_0^2, t_0^3) \in Q^3$, let

$$+ \frac{1}{2} Q_j^3(x) = \{(t^1, t^2, t^3) \in Q^3 : t^j \geq t_0^j\}$$

and

$$- \frac{1}{2} Q_j^3(x) = \{(t^1, t^2, t^3) \in Q^3 : t^j \leq t_0^j\}, \quad j = 1, 2, 3.$$

We shall call the following pairs (x, F) of points $x \in Q^3$ and closed subsets $F \ni x^*$ in Q^3 marked:

- a) $x = (i', i'', i'''), F = Q^3$;
- b) $x = (k, t', t'')^{**}$, t' and $t'' \in R_{\bar{k}} \cup I$, $F = Q^3$, $k = 0, 1$;
- c) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^j \in R$, $t^{j'} \in I$ for $j' \neq j$, $F = +\frac{1}{2}Q_j^3(x)$;
- d) $x = (k, r_k, t)$, $t \in R_{\bar{k}} \cup I$, $F = +\frac{1}{2}Q_2^3(x)$, $k = 0, 1$;
- $x = (k, t, r_k)$, $t \in R_{\bar{k}} \cup I$, $F = +\frac{1}{2}Q_3^3(x)$, $k = 0, 1$;
- (1) e) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^{j'}$ and $t^{j''} \in R$, $j' \neq j''$,
and $t^j \in I$ for $j \neq j'$ and $j \neq j''$, $F = +\frac{1}{2}Q_{j'}^3(x) \cap +\frac{1}{2}Q_{j''}^3(x)$;
- f) $x = (k, r_k, r'_k)$, $F = +\frac{1}{2}Q_2^3(x) \cap +\frac{1}{2}Q_3^3(x)$, $k = 0, 1$;
- g) $x = (r^1, r^2, r^3)$, $0 < r^1 < 1$, $F = \bigcap_{j=1}^3 +\frac{1}{2}Q_j^3(x)$.

* To shorten the notation we shall adopt the following convention. If a pair is marked

1) $(x, +\frac{1}{2}Q_j^3(x))$, or 2) $(x, +\frac{1}{2}Q_{j'}^3(x) \cap +\frac{1}{2}Q_{j''}^3(x))$, or 3) $(x, +\frac{1}{2}Q_{j'}^3(x) \cup +\frac{1}{2}Q_{j''}^3(x))$, or 4) $(x, \bigcap_{j=1}^3 +\frac{1}{2}Q_j^3(x))$, or 5) $(x, +\frac{1}{2}Q_{j'''}^3(x) \cup +\frac{1}{2}Q_{j''}^3(x) \cap +\frac{1}{2}Q_{j'}^3(x))$, then in each of the cases 1)–5) all pairs obtained from the indicated pair by replacing, in its second element, some (or all) plus signs by minus signs are also regarded as marked. The indices j, j', j'', j''' may take the values 1, 2, 3.

** Everywhere k is equal either to 0 or to 1 and, if $k = 0$, then $\bar{k} = 1$, while if $k = 1$, then $\bar{k} = 0$.

- (2) h) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^{i'} \in R_0$, $t^{i''} \in R_1$, $j' \neq j''$, $t^j \in I$ for $j \neq j'$ and $j \neq j''$,

$$F = +^{1/2}Q_{j'}^3(x) \cup +^{1/2}Q_{j''}^3(x);$$

- i) $x = (t^1, t^2, t^3)$, $0 < t^1 < 1$, $t^{j'}$ and $t^{j''} \in R_k$, $j' \neq j''$, $t^{j'''} \in R_{\bar{k}}$, $j''' \neq j'$ and $j''' \neq j''$,

$$F = +^{1/2}Q_{j'''}^3(x) \cup (+^{1/2}Q_{j'}^3(x) \cap +^{1/2}Q_{j''}^3(x)).$$

Obviously, the cardinality of the set B of all marked pairs is $\mathbf{c}$. The set of the pairs marked in items a)–g) will be denoted by B_1 , and the set of the pairs marked in items h) and i), by B_2 .

- 3. Let a bicom pactum P_0 be contained in a bicom pactum P_1 and have a continuous mapping f onto a bicom pactum P . As the elements of the decomposition τ of the bicom pactum P_1 we take the points of the set $P_1 \setminus P_0$ and the inverse images of points of the bicom pactum P under the mapping f . We shall say that the bicom pactum P_1^* , which is the

decomposition space τ , is obtained from the bicom pactum P_1 by gluing the bicom pactum P_0 to the bicom pactum P by means of the mapping f .

4. Let a bicom pactum X and a system of pairs (x_θ, F_θ) , $\theta \in \Theta$, of points $x_\theta \in X$ and closed subsets $F_\theta \ni x_\theta$ in X be given. Let there also be given a bicom pactum χ and in it a disjoint system of open sets C_θ , $\theta \in \Theta$, each of which decomposes into the disjoint sum of bicom pacta C_α , $\alpha \in \mathfrak{A}_\theta$, open-and-closed in χ . Suppose that for each α a mapping $f_\alpha : C_\alpha \rightarrow \Phi_\alpha$ is defined. In the bicom pactum

$$X \times \chi \setminus \bigcup_{\theta \in \Theta} \bigcup_{\alpha \in \mathfrak{A}_\theta} (X \setminus F_\theta \times C_\alpha)$$

for each $\alpha \in \mathfrak{A}_\theta$ we glue the bicom pactum $x_\theta \times C_\alpha$ to the bicom pactum Φ_α by means of the mapping f_α , $\theta \in \Theta$. The resulting bicom pactum will be denoted by

$$D(X, \chi, \{x_\theta, F_\theta, C_\alpha, \Phi_\alpha, f_\alpha; \alpha \in \mathfrak{A}_\theta\}, \theta \in \Theta).$$

5. By the tail χ_1 we shall mean the ordered zero-dimensional bicom pactum obtained as follows. Take the set

$$\chi_1 = \bigcup_{\alpha \leq \omega(\mathfrak{c})} \alpha \cup \bigcup_{\alpha < \omega(\mathfrak{c})} C_\alpha,$$

where $\omega(\mathfrak{c})$ denotes the first ordinal of cardinality $\mathfrak{c}$, α are ordinals, and C_α are Cantor perfect sets.

The sets C_α are taken in their natural topology and are regarded as open-and-closed in χ_1 ; as a basic neighborhood of a nonlimit number α we take the number itself, and as basic neighborhoods of a limit number α' we take sets of the form

$$\bigcup_{\alpha'' < \alpha \leq \alpha'} \alpha \cup \bigcup_{\alpha'' < \alpha < \alpha'} C_\alpha.$$

Represent the set

$$\mathfrak{A} = \{\alpha < \omega(\mathfrak{c})\}$$

as a disjoint sum of sets $\mathfrak{A}_\theta$, $\theta \in \Theta$, of cardinality $\mathfrak{c}$, so that the cardinality of Θ is also equal to $\mathfrak{c}$.

Between the sets Θ and B (see item 2) we establish a one-to-one correspondence.

6. We now construct a bicom pactum T^3 with

$$\dim T^3 = \text{ind } T^3 = 3 < \text{Ind } T^3 = 4.$$

Consider the product $Q^3 \times \chi_1$. Fix some mapping f of the Cantor set C onto Q^3 and some mapping g of the Cantor set C onto the square Q^2 . Then

$$T^3 = D(Q^3, \chi_1, \{x_\theta, F_\theta, C_\alpha, \Phi_\alpha, f_\alpha; \alpha \in \mathfrak{A}_\theta\}, \theta \in \Theta),$$

where (x_θ, F_θ) is the element of the set B (i.e. the marked pair) corresponding to the element θ of the set Θ ; moreover, if $(x_\theta, F_\theta) \in B_1$, then $\Phi_\alpha = Q^3$ and $f_\alpha = f$, while if $(x_\theta, F_\theta) \in B_2$, then $\Phi_\alpha = Q^2$ and $f_\alpha = g$.

The bicom pactum T^3 has cardinality $\mathfrak{c}$ and decomposes into the sum of the cube $Q^3 \times \omega(\mathfrak{c})$ and an additional set which is locally metrizable if $\mathfrak{c} = \aleph_1$. Let us also note that the set

$$T^3 \setminus Q^3 \times \{\alpha \leq \omega(\mathfrak{c})\}$$

is metrizable.

5'. By the tail χ_2 we shall mean the ordered zero-dimensional bicom pactum with the first axiom of countability, obtained as follows.

Consider the lexicographically ordered product

$$U = Q^1 \times D$$

of the interval $Q^1 = [0, 1] = \{u\}$ and the pair of isolated points

$$D = \{0, 1\}.$$

This bicom pactum was constructed in (4) and is called "two arrows." Let $u_\ell = (u, 0)$, $u_p = (u, 1)$. Put in correspondence with each number u

the lexicographically ordered product $V_u = Q_u^1 \times D$ of the interval $Q_u^1 = [0, 1] = \{v_u\}$ and the pair of isolated points $D = \{0, 1\}$. Let $v_{u\ell} = (v_u, 0)$ and $v_{ur} = (v_u, 1)$. Finally, to each number $v_u \in Q_u^1$, $u \in [0, 1]$, we assign the Cantor set C_{v_u} . Then

$$\chi_2 = U \cup \bigcup_{0 \leq u \leq 1} V_u \cup \bigcup_{\substack{0 \leq u \leq 1 \\ 0 \leq v_u \leq 1}} C_{v_u}.$$

The elements of the sets C_{v_u} , V_u , and U have already been ordered. We put $v_{u\ell} < c_{v_u} < v_{ur}$ for any point $c_{v_u} \in C_{v_u}$, and $u_\ell < v_u < u_r$ for any point $v_u \in V_u$. The order relation for the other pairs of points is defined by transitivity. The order (interval) topology gives the required bicom pactum χ_2 . Represent the interval $Q_u^1 = [0, 1] = \{v_u\}$ as a disjoint sum of everywhere dense sets $\mathfrak{A}_{\theta u} = \{a = v_u\}$, $\theta \in \Theta$, such that the cardinality of Θ is $\mathfrak{c}$, $0 \leq u \leq 1$. Let $\mathfrak{A}_\theta = \bigcup_{0 \leq u \leq 1} \mathfrak{A}_{\theta u}$, $\theta \in \Theta$. Between the sets Θ and B establish a one-to-one correspondence.

6'. We now construct a bicom pactum S^3 with the first axiom of countability and with

$$\dim S^3 = \text{ind } S^3 = 3 < \text{Ind } S^3 = 4.$$

Consider the product $Q^3 \times \chi_2$. Then

$$S^3 = D(Q^3, \chi_2, \{x_\theta, F_\theta, C_\alpha, \Phi_\alpha, f_\alpha, \alpha \in \mathfrak{A}_\theta\}, \theta \in \Theta),$$

where (x_θ, F_θ) is the element of the set B (i.e. the marked pair) corresponding to the element θ of the set Θ ; moreover, if $(x_\theta, F_\theta) \in B_1$, then $\Phi_\alpha = Q^3$ and $f_\alpha = f$ (see item 6), while if $(x_\theta, F_\theta) \in B_2$, then $\Phi_\alpha = Q^2$ and $f_\alpha = g$.

The bicom pactum S^3 decomposes into the sum: a) of the product $Q^3 \times (U \cup \bigcup_u V_u)$ of the cube and a zero-dimensional ordered bicom pactum with the first axiom of countability, and b) of a metrizable complement to this product*.

7. The method of constructing the bicom pacta T^3 and S^3 is of a general nature.

Theorem. For any $n \geq 3$ there exist bicom pacta T^n and S^n of cardinality $\mathfrak{c}$ and dimension

$$\dim T^n = \text{ind } T^n = \dim S^n = \text{ind } T^n = n < \text{Ind } T^n = \text{Ind } S^n = n + 1,$$

- 1) The bicom pactum T^n decomposes into the sum: a) of the product of the n -dimensional cube Q^n and the ordinal numbers $\leq \omega(\mathfrak{c})$, b) of a metrizable complement to this product*. If $\mathfrak{c} = \aleph_1$, then the complement to the cube $Q^n \times \omega(\mathfrak{c})$ is locally metrizable.
- 2) The bicom pactum S^n has the first axiom of countability and decomposes into the sum: a) of the product of the n -dimensional cube Q^n and a zero-dimensional ordered bicom pactum with the first axiom of countability, b) of a metrizable complement to this product*.

Corollary. There exist locally bicom pact metrizable spaces of weight and cardinality $\mathfrak{c}$ having bicom pact extensions (even with the first axiom of countability) with different inductive dimensions.

Mechanics and Mathematics Faculty
M. V. Lomonosov Moscow State University

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CITED LITERATURE

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- ⁴ P. S. Aleksandrov, P. S. Uryson, in the book: P. S. Uryson, *Works on Topology and Other Fields of Mathematics*, 2, Moscow-Leningrad, 1951, p. 848.

* Which is a discrete sum of compacta.

Note: Figure translations are in progress. See original paper for figures.

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