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Abstract

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MATHEMATICS

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ON ESTIMATING THE BEST APPROXIMATION OF AN INTEGRABLE FUNCTION ON THE REAL AXIS BY ENTIRE FUNCTIONS OF FINITE DEGREE

Let $W_\sigma^{(p)} (p \geq 1)$ be the class of entire functions $g_\sigma(z)$ of finite degree $\leq \sigma$, for which $g_\sigma(x) \in L_p(-\infty, \infty)$, $p \geq 1$. Denote by $A_\sigma(f)_p$ the best approximation of a function $f(x) \in L_p(-\infty, \infty)$ in the metric of the space $L_p(-\infty, \infty)$ by entire functions from the class $W_\sigma^{(p)}$, i.e.

$$A_\sigma(f)_p = \inf_{g_\sigma \in W_\sigma^{(p)}} \left(\int_{-\infty}^{\infty} |f(x) - g_\sigma(x)|^p dx \right)^{1/p},$$

and introduce for consideration the moduli of continuity

$$\omega_1(f; \delta)_p = \sup_{|h| \leq \delta} \left(\int_{-\infty}^{\infty} |f(x+h) - f(x)|^p dx \right)^{1/p},$$

$$\omega_2(f; \delta)_p = \sup_{|h| \leq \delta} \left(\int_{-\infty}^{\infty} |f(x+h) - 2f(x) + f(x-h)|^p dx \right)^{1/p}.$$

It is known that for a function $f(x) \in L_p(-\infty, \infty)$ there exist constants K_1 and K_2 satisfying the inequalities:

$$A_\sigma(f)_p \leq K_i \omega_i(f; \delta)_p \quad (i = 1, 2),$$

which are analogues of the classical Jackson theorem for periodic functions (see ⁽¹⁾, p. 274).

The present article is devoted* to finding the possible least values K_i^0 of the constants K_i ($i = 1, 2$), mainly for $p = 2$. First of all, two auxiliary lemmas are proved.

Lemma 1. Let $f(x) \in L_2(-\infty, \infty)$, and let $\varphi(x)$ be its Fourier transform in the sense of $L_2(-\infty, \infty)$, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \varphi(t) \frac{e^{ixt} - 1}{it} dt, \quad (1)$$

where $\varphi(x) \in L_2(-\infty, \infty)$. Then the function

$$g_{\sigma}^0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \varphi(t) e^{ixt} dt \quad (2)$$

is an entire function from the class $W_{\sigma}^{(2)}$, least deviating from $f(x)$ in the metric $L_2(-\infty, \infty)$. Moreover,

$$A_{\sigma}(f_2) = \|f(x) - g_{\sigma}^0(x)\|_2 = \left(\int_{|t|>\sigma} |\varphi(t)|^2 dt \right)^{1/2}. \quad (3)$$

Indeed, applying the known Plancherel theorem to the function

$$f(x) - g_{\sigma}(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} [\varphi(x) - \psi(x)] \frac{e^{ixt} - 1}{it} dt,$$

* Works ⁽²⁻⁵⁾ are devoted to similar questions in the periodic case.

where $\psi(t) = 0$ for $|t| > \sigma$, we find

$$\inf_{g_{\sigma} \in W_{\sigma}^{(2)}} \|f(x) - g_{\sigma}(x)\|_2^2 = \inf_{\psi \in L_2(-\sigma, \sigma)} \int_{-\sigma}^{\sigma} |\varphi(t) - \psi(t)|^2 dt + \int_{|t|>\sigma} |\varphi(t)|^2 dt.$$

This implies equality (3). Applying Lemma 1 to the function

$$f^{(r)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{ixt} - 1}{it} (it)^r \varphi(t) dt \in L_2(-\infty, \infty),$$

we find

$$A_{\sigma}(f^{(r)})_2 = \left\{ \int_{|t|>\sigma} |(it)^r \varphi(t)|^2 dt \right\}^{1/2} > \sigma^r \left\{ \int_{|t|>\sigma} |\varphi(t)|^2 dt \right\}^{1/2} = \sigma^r A_{\sigma}(f)_2.$$

Lemma 2. If $\varphi(t)$ is any function from $L_2(-\infty, \infty)$, then

$$\inf_{|y| \leq \pi/\sigma} \int_{|t| > \sigma} |\varphi(t)|^2 \cos yt \, dt < 0. \quad (4)$$

Proof. Extending evenly to $(-\infty, 0)$ the function

$$\varphi_\sigma(y) = -\sin \sigma y \quad \text{for } 0 \leq y \leq \pi/\sigma; \quad \varphi_\sigma(y) = 0 \quad \text{for } y > \pi/\sigma,$$

we represent it in the form of a Fourier integral

$$\varphi_\sigma(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{\varphi}_\sigma(t) \cos yt \, dt.$$

From this we find

$$\tilde{\varphi}(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi_\sigma(y) \cos yt \, dt = \frac{1}{\sqrt{2\pi}} \frac{4\sigma}{t^2 - \sigma^2} \cos^2 \frac{\pi t}{2\sigma}.$$

Let us note that the function

$$F_\sigma(y) = \sqrt{\pi/2} |\varphi(t)|^2, \quad |t| > \sigma; \quad F_\sigma(y) = 0, \quad |t| \leq \sigma,$$

is the cosine transform of the function

$$F_\sigma(y) = \int_\sigma^{+\infty} |\varphi(t)|^2 \cos yt \, dt, \quad 0 \leq y \leq \pi/\sigma.$$

It is not difficult to show that

$$\int_0^\infty F_\sigma(y) \varphi_\sigma(y) \, dy = - \int_0^\infty \tilde{F}_\sigma(y) \tilde{\varphi}_\sigma(y) \, dy = 2\sigma \int_\sigma^\infty |\varphi(t)|^2 \frac{\cos^2 \pi t / 2\sigma}{t^2 - \sigma^2} \, dt \geq 0. \quad (5)$$

On the other hand, if $F_\sigma(y) \geq 0$ everywhere on $[0, \pi/\sigma]$, then

$$\int_0^\infty F_\sigma(y) \varphi_\sigma(y) \, dy = - \int_0^{\pi/\sigma} F_\sigma(y) \sin \sigma y \, dy < 0. \quad (6)$$

Inequality (6) contradicts (5), and, consequently, $F_\sigma(y)$ assumes both positive and negative values on $[0, \pi/\sigma]$, i.e. (4) is true. Owing to these lemmas, the following is proved.

Theorem 1. If the function $f(x) \in L_2(-\infty, \infty)$, then for the best approximation $A_\sigma(f)_2$ the inequalities

$$A_\sigma(f)_2 < \frac{1}{\sqrt{2}} \omega_1(f; \pi/\sigma)_2, \quad (7)$$

$$A_\sigma(f)_2 < \frac{1}{2} \omega_2(f; \pi/\sigma)_2 \quad (8)$$

hold.

Proof. From equality (1) we find

$$\omega_1\left(f; \frac{\pi}{\sigma}\right)_2 = \sup_{|y| \leq \pi/\sigma} \|f(x+y) - f(x)\|_2 = \sqrt{2} \left\{ \int_{|t| > \sigma} |\varphi(t)|^2 dt - \inf_{0 \leq y \leq \pi/\sigma} F_\sigma(y) \right\}.$$

Hence, by Lemmas 1 and 2, (7) follows. Moreover, for $|t| \leq \pi/\sigma$,

$$\begin{aligned} \sup_{|t| \leq \pi/\sigma} \|f(x+t) - 2f(x) + f(x-t)\|_2^2 &\geq 4 \sup_{|t| \leq \pi/\sigma} \int_{|u| > \sigma} |\varphi(u)|^2 (1 - \cos ut)^2 du \geq \\ &\geq 4 \int_{|u| > \sigma} |\varphi(u)|^2 du - 8 \inf_{0 \leq y \leq \pi/\sigma} F_\sigma(y). \end{aligned}$$

Hence inequality (8) follows.

Corollary. If $f(x)$ has a derivative $f^{(r)}(x) \in L_2(-\infty, \infty)$, then the estimates

$$A_\sigma(f)_2 \leq \frac{1}{\sqrt{2}\sigma^r} \omega_1(f^{(r)}; \pi/\sigma)_2, \quad A_\sigma(f)_2 \leq \frac{1}{2\sigma^r} \omega_2(f^{(r)}; \pi/\sigma)_2$$

are valid.

Along the way it is established that, for the best approximation of an entire function

$$f_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \varphi(t) e^{ixt} dt$$

of degree λ , where $\varphi(t) \in L_2(-\lambda, \lambda)$, by entire functions from the class $W_\sigma^{(2)}$, we have

$$A_{\sigma}(f_{\lambda})_2 = \left(\int_{\sigma < |t| \leq \lambda} |\varphi(t)|^2 dt \right)^{1/2}.$$

As an example, the function $f(x) = e^{-|x|} \in L_2(-\infty, \infty)$ with Fourier transform

$$\varphi(t) = \sqrt{\frac{2}{\pi}} \frac{1}{1+t^2}$$

is considered. According to Lemma 1, we have

$$A_{\sigma}(e^{-|x|})_2 = \left[1 - \frac{2}{\pi} \left(\arctg \sigma - \frac{\sigma}{1+\sigma^2} \right) \right]^{1/2} < \sqrt{\frac{2}{\sigma}}.$$

Now let $f(x)$ be an arbitrary function from the class $L_p(-\infty, \infty)$ ($1 < p \leq 2$). Then

$$F(x, a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a f(t) e^{ixt} dt$$

as $a \rightarrow \infty$ converges in the mean with exponent q , where $q = p/(p-1)$. The mean limit $F(x)$, called the Fourier transform of the function $f(x)$, satisfies the inequality ⁽⁶⁾

$$\left(\int_{-\infty}^{\infty} |F(x)|^q dx \right)^{1/q} \leq B_{p,q} \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p},$$

where

$$B_{p,q} = (2\pi/q)^{1/2q} (p/2\pi)^{1/2p}.$$

Moreover, for almost all x we have

$$F(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} f(t) \frac{e^{ixt} - 1}{it} dt, \quad f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} F(t) \frac{e^{-itx} - 1}{-it} dt.$$

Thanks to these assertions, the following is proved.

Theorem 2. If $f(x) \in L_p(-\infty, \infty)$ ($1 < p \leq 2$) and $f(x)$ is its Fourier transform in the sense of $L_p(-\infty, \infty)$, then the estimates

$$A_{\sigma}(f)_p \geq B_{p,q}^{-1} \left(\int_{|x|>\sigma} |F(x)|^q dx \right)^{1/q} \quad (1/p + 1/q = 1),$$

$$\sup_{|y| \leq \pi/\sigma} \left(\int_{-\infty}^{\infty} |F(x)|^q \left| \sin \frac{yx}{2} \right|^q dx \right)^{1/q} \leq \frac{1}{2} B_{p,q} \omega_1 \left(f; \frac{\pi}{\sigma} \right)_p.$$

The arguments used in proving the assertions given above can also be applied in the multidimensional case. For example, consider a function admitting the representation

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixt+iy\tau} \varphi(t, \tau) dt d\tau, \quad (9)$$

and entire functions $g_{\sigma_1, \sigma_2}(x, y) \in W_{\sigma_1, \sigma_2}^{(p)}$, admitting the spectral representation

$$g_{\sigma_1, \sigma_2}(x, y) = \frac{1}{2\pi} \int_{-\sigma_1}^{\sigma_1} \int_{-\sigma_2}^{\sigma_2} \psi(t, \tau) e^{ixt+iy\tau} dt d\tau,$$

where $\psi(t, \tau) \in L_2(-\sigma_1, \sigma_1; -\sigma_2, \sigma_2)$ and $\psi(t, \tau) = 0$ for $(x, y) \notin \left[-\sigma_1, \sigma_1; -\sigma_2, \sigma_2 \right]$.

Theorem 3. If the function $f(x, y) \in L_2(-\infty, \infty)$ is defined by equality (9), then the entire function from the class $W_{\sigma_1, \sigma_2}^{(2)}$ least deviating from it in the sense of the metric $L_2(-\infty, \infty)$ is

$$g_{\sigma_1, \sigma_2}^0(x, y) = \frac{1}{2\pi} \int_{-\sigma_1}^{\sigma_1} \int_{-\sigma_2}^{\sigma_2} e^{ixt+iy\tau} \varphi(t, \tau) dt d\tau, \quad (10)$$

where $\varphi(x, y)$ is the Fourier transform of the function $f(x, y)$ in the sense of (9); moreover, the value of the best approximation $A_{\sigma_1, \sigma_2}(f)_2$ of the function $f(x, y)$ by entire functions from the class $W_{\sigma_1, \sigma_2}^{(2)}$ is computed from one of the equalities

$$A_{\sigma_1, \sigma_2}(f)_2 = \left\{ \int_{-\infty}^{\infty} \int_{|\tau| \geq \sigma_2} |\varphi(t, \tau)|^2 dt d\tau + \int_{|t| > \sigma_1} \int_{|\tau| \leq \sigma_2} |\varphi(t, \tau)|^2 dt d\tau \right\}^{1/2}, \quad (11)$$

or

$$A_{\sigma_1, \sigma_2}(f)_2 = \left\{ \int_{|t| > \sigma_1} \int_{-\infty}^{\infty} |\varphi(t, \tau)|^2 dt d\tau + \int_{|t| \leq \sigma_1} \int_{|\tau| > \sigma_2} |\varphi(t, \tau)|^2 dt d\tau \right\}^{1/2}. \quad (12)$$

Introduce the notation:

$$A_{\sigma_1, \infty}(f)_p = \inf_{g_{\sigma_1, \infty} \in W_{\sigma_1, \infty}^{(p)}} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y) - g_{\sigma_1, \infty}(x, y)|^p dx dy \right\}^{1/p} \quad (p \geq 1),$$

where

$$g_{\sigma_1, \infty}(x, y) = \int_{-\sigma_1}^{\sigma_1} \int_{-\infty}^{\infty} e^{ixt + iy\tau} \varphi(t, \tau) dt d\tau$$

is an entire function in x from the class $W_{\sigma_1}^{(p)}$. The quantity $A_{\infty, \sigma_2}(f)_p$ is defined analogously, and it is proved that

$$A_{\sigma_1, \sigma_2}(f)_2 \leq [A_{\sigma_1, \infty}^2(f)_2 + A_{\infty, \sigma_2}^2(f)_2]^{1/2}.$$

Owing to this, from the inequalities

$$A_{\sigma_1, \infty}^2(f)_2 < \frac{1}{2} \omega_1^2 \left(f; \frac{\pi}{\sigma_1}; 0 \right)_2; \quad A_{\infty, \sigma_2}^2(f)_2 < \frac{1}{2} \omega_1^2 \left(f; 0; \frac{\pi}{\sigma_2} \right)_2$$

we obtain

$$A_{\sigma_1, \sigma_2}(f)_2 < \frac{1}{\sqrt{2}} \left\{ \omega_1^2 \left(f; \frac{\pi}{\sigma_1}; 0 \right)_2 + \omega_1^2 \left(f; 0; \frac{\pi}{\sigma_2} \right)_2 \right\}^{1/2}.$$

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Note: Figure translations are in progress. See original paper for figures.

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