

ON THE ROLE OF THE EXTREME TERMS OF A VARIATION SERIES IN THE FORMATION OF A LARGE DEVIATION OF THE SUM OF INDEPENDENT RANDOM VARIABLES

A. V. Nagaev

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Abstract

Full Text

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MATHEMATICS

A. V. NAGAEV

ON THE ROLE OF THE EXTREME TERMS OF A VARIATION SERIES IN THE FORMATION OF A LARGE DEVIATION OF THE SUM OF INDEPENDENT RANDOM VARIABLES

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Let ξ_j , $j = 1, 2, \dots$, be independent identically distributed random variables, $M\xi_j = 0$, $D\xi_j = \sigma^2 < \infty$. Let, further,

$$\underline{\xi} = \xi_1^* \leq \xi_2^* \leq \dots \leq \xi_n^* = \bar{\xi} \quad (1)$$

be the variation series constructed from the realization $\xi_1, \xi_2, \dots, \xi_n$. We shall be interested in the limiting distribution law of the extreme terms of the series (1) under conditions imposed on the sum $\zeta_n = \xi_1 + \dots + \xi_n$.

At first we shall assume that the random variables ξ_j have an absolutely continuous distribution with bounded density $p(x)$, and that, as $x \rightarrow 0$,

$$p(x) \sim e^{-x^\alpha}, \quad \alpha > 0. \quad (2)$$

Put, as usual,

$$\Phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^w e^{-u^2/2} du.$$

Let, further, the sequences A_n and B_n satisfy the equalities*

$$2n \exp\{-A_n^2/2\} = A_n \sqrt{2\pi}, \quad B_n = A_n^{-1}, \quad (3)$$

and let the sequences a_n and b_n be such that

$$n \exp\{-a_n^\alpha\} = \alpha a_n^{\alpha-1}, \quad b_n = \alpha a_n^{1-\alpha}. \quad (4)$$

Theorem 1. Let in relation (2) $\alpha > 1$. Then

1°. If $0 \leq x \leq n(\ln n)^{-\gamma}$, $\gamma > 1/\alpha$, then as $n \rightarrow \infty$

$$\mathbf{P}\{\bar{\xi} < a_n + b_{nz} \mid \zeta_n = x\} = \mathbf{P}\{\bar{\xi} < a_n + b_{nz}\} + o(1) = \exp\{-e^{-z}\} + o(1)$$

uniformly in z , $z \geq z_0 > -\infty$.

2°. If, however, $x \geq n(\ln n)^{\gamma_1}$, $\gamma_1 > \max(12/\alpha, 1/(\alpha - 1))$, then as $n \rightarrow \infty$

$$\begin{aligned} \mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\xi_j - \frac{x}{n}\right| \sqrt{\frac{x^{\alpha-2}}{\alpha(\alpha-1)}} < A_n + B_{nz} \mid \zeta_n = x\right\} = \\ = \exp\{-e^{-z}\} + o(1) \end{aligned}$$

uniformly in z , $z \geq z_0 > -\infty$.

* It is easy to see that, uniformly in z , $-\infty < z < \infty$,

$$\mathbf{P}\left\{\max_{1 \leq j \leq n} |\eta_j| < A_n + B_{nz}\right\} = \exp\{-e^{-z}\} + o(1),$$

where the η_j are independent and normally distributed with parameters $(0, 1)$.

Here the sequences A_n, B_n, a_n , and b_n are defined by equations (3) and (4).

Theorem 2. Suppose that in relation (2) $0 < \alpha < 1$. Suppose further that $M|\xi_1|^k < \infty$, $k = 4 + \left[\frac{2\alpha-1}{1-\alpha}\right]$ (in particular, for $0 < \alpha < 1/2$ we have $k = 3$). Then:

1°. If

$$0 \leq x \leq (c_\alpha - \delta)\sigma^{2/(2-\alpha)}n^{1/(2-\alpha)}, \quad c_\alpha = (2-\alpha)(2-2\alpha)^{(\alpha-1)/(2-\alpha)},$$

where $\delta > 0$ is an arbitrarily small positive number, then, as $n \rightarrow \infty$,

$$\mathbf{P}\{\bar{\xi} < a_n + b_{nz} \mid \zeta_n = x\} = \mathbf{P}\{\bar{\xi} < a_n + b_{nz}\} + o(1) = \exp\{-e^{-z}\} + o(1)$$

uniformly in z , $z \geq z_0 > -\infty$, and w , $-\infty < w < \infty$.

2°. If, however,

$$(c_\alpha - \delta)\sigma^{2/(2-\alpha)}n^{1/(2-\alpha)} < x = o(n^{1/(2-2\alpha)}),$$

then, as $n \rightarrow \infty$,

$$\mathbf{P}\{\xi_{n-1}^* < a_n + b_{nz}; \bar{\xi} - (1-\beta)x < w\sigma\sqrt{n} \mid \zeta_n = x\} =$$

$$= \exp\{-e^{-z}\}\Phi\left(\frac{w}{\sigma_1}\right) + o(1)$$

uniformly in z , $z \geq z_0 > -\infty$, and w , $-\infty < w < \infty$.

Here the quantity β is the smaller positive root of the equation

$$\frac{n\sigma^2}{x^{2-\alpha}} = \frac{\beta(1-\beta)^{1-\alpha}}{\alpha},$$

if $0 < \alpha < 1/2$, and of the equation

$$\frac{n\sigma^2}{x^{2-\alpha}} = \frac{\beta(1-\beta)^{1-\alpha}}{\alpha} \left[1 - \sigma^2 \sum_{j=0}^{k-4} (j+3)\lambda_j \left(\frac{\beta x}{n}\right)^{j+1} \right],$$

if $1/2 \leq \alpha < 1$.

Here the sequences a_n and b_n are defined by equalities (4), while λ_j , $j = 0, \dots, k-4$, are the first coefficients of the Cramér series (see ⁽¹⁾, equality (19)), and

$$\sigma_1 = (1 - \alpha(1 - \alpha)n\sigma^2 / (1 - \beta)^{1+\varepsilon} x^{1+\varepsilon})^{-1/2}.$$

Theorem 3. Suppose that in relation (2) $0 < \alpha < 1$. Then:

1°. If $n^{1/(2-2\alpha)} \ll x$, then, as $n \rightarrow \infty$,

$$\begin{aligned} \mathbf{P}\{\xi_{n-1}^* < a_n + b_{nz}; \bar{\xi} - x + n\alpha\sigma^2 x^{\alpha-1} < w\sigma\sqrt{n} \mid \zeta_n = x\} = \\ = \exp\{-e^{-z}\}\Phi(w) + o(1) \end{aligned}$$

uniformly in z , $z \geq z_0 > -\infty$, and w , $-\infty < w < \infty$.

2°. If, however, $n^{1/(2-2\alpha)} = o(x)$, then, as $n \rightarrow \infty$,

$$\begin{aligned} \mathbf{P}\{\xi_{n-1}^* < a_n + b_{nz}; \bar{\xi} < x + w\sigma\sqrt{n} \mid \zeta_n = x\} = \\ = \exp\{-e^{-z}\}\Phi(w) + o(1) \end{aligned}$$

uniformly in z , $z \geq z_0 > -\infty$, and w , $-\infty < w < \infty$.

The theorems stated above refine the considerations on the nature of large deviations contained in the papers ^(2,3). We now briefly describe the case where the random variables are integer-valued. Suppose that instead of representation (2) the relation

$$\mathbf{P}\{\xi_1 = k\} \sim e^{-k^\alpha}, \quad \alpha > 0. \quad (5)$$

holds.

Under condition (5), the assertions of Theorems 2 and 3 remain valid. If in relation (5) $1 < \alpha < 2$, then assertion 2° of Theorem 1 is also valid. Under the conditions of 1° of Theorem 1, there is not even an unconditional limiting distribution for $\bar{\xi}$. Concerning the case $\alpha \geq 2$, see paper ⁽⁴⁾.

If one does not require the existence of a density of the distribution of the random variables ξ_j and does not assume that they are integer-valued, then one has to use

to trace the limiting behavior of our order statistics under the condition that $\zeta_n > x$. For example, the following is valid.

Theorem 4. *Suppose that instead of representation (2) the relation*

$$1 - F(x) \sim x^{-\gamma}, \quad \gamma > 2. \quad (6)$$

holds. Then, as $n \rightarrow \infty$, $x/\ln x \geq \sqrt{n}$

$$\mathbf{P}\{\xi_{n-1}^* < zn^{1/\gamma}; \bar{\xi} > x + wx \mid \zeta_n > x\} = \exp\{-z^{-\gamma}\}(1+w)^{-\gamma} + o(1)$$

uniformly in z , $z \geq z_0 > -\infty$, and w , $w \geq 0$.

Conditions (2), (5), and (6) are one-sided in character; therefore the theorems given above (except for item 2° of Theorem 1) describe the limiting law of distribution only for the extreme right-hand terms of series (1). If the behavior of the probability $\mathbf{P}\{\xi_1 < x\}$ as $x \rightarrow -\infty$ is specified, then one can obtain an explicit expression for the limiting distribution of the statistics ξ , $\bar{\xi}$, and ξ_{n-1}^* . For example, instead of item 2° of Theorem 3 the following may be formulated.

Theorem 5. *Suppose that in representation (2) $0 < \alpha < 1$, and as $x \rightarrow \infty$*

$$\mathbf{P}\{\xi_1 < -x\} \sim x^{-\gamma}, \quad \gamma > 4 + \left[\frac{2\alpha - 1}{1 - \alpha} \right].$$

Then, if $n^{1/(2-2\alpha)} = o(x)$,

$$\mathbf{P}\{-yn^{1/\gamma} < \xi; \xi_{n-1}^* < a_n + b_{nz}; \bar{\xi} < x + w\sigma\sqrt{n} \mid \zeta_n = x\} = \exp\{-y^{-\gamma} - e^{-z}\}\Phi(w) + o(1)$$

uniformly in y , z , and w , $y \geq 0$, $z \geq z_0 > -\infty$, $-\infty < w < \infty$.

Romanovsky Institute of Mathematics
Academy of Sciences of the Uzbek SSR
Tashkent

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Note: Figure translations are in progress. See original paper for figures.

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