

LIMITING DISTRIBUTIONS OF FUNCTIONALS OF A SEMI-MARKOV PROCESS DEFINED ON A FIXED SET OF STATES UP TO THE MOMENT OF FIRST EXIT

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Abstract

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MATHEMATICS

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LIMITING DISTRIBUTIONS OF FUNCTIONALS OF A SEMI-MARKOV PROCESS DEFINED ON A FIXED SET OF STATES UP TO THE MOMENT OF FIRST EXIT

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Consider, for each $t \in (0, \infty)$, a right-continuous semi-Markov process (S.M.P.) $\varkappa_t(s) \in \{0, 1, \dots, r\}$, which is specified, following ^(1,2), by the matrix of transition probabilities

$$F_t(i, j, u) = P\{\varepsilon_{k+1} = j, \tau_t(\varepsilon_k) < u \mid \varepsilon_k = i\}, \quad i, j = 0, \dots, r,$$

where $\varepsilon_k = \varkappa_t(\theta_t(k))$, $\tau_t(\varepsilon_k) = \theta_t(k+1) - \theta_t(k)$, and $\theta_t(k)$ is the moment of the k -th jump, i.e. $\theta_t(0) = 0$, and

$$\theta_t(k) = \min\{s : s > \theta_t(k-1), \varkappa_t(s) \neq \varepsilon_{k-1}\}, \quad k \geq 1.$$

We shall call the set of states $\langle \omega \rangle = \langle 1, 2, \dots, r \rangle$ a group if, for any $i, j \in \langle \omega \rangle$, $p_t(i, j, \langle \omega \rangle) \rightarrow 1$ as $t \rightarrow \infty$, where $p_t(i, j, \langle \omega \rangle)$ is the probability of reaching j from i before the S.M.P. leaves $\langle \omega \rangle$.

Let, for each $t \in (0, \infty)$, $f_t^{(k)}(i, x)$, $k = 0, 1, \dots$, $i \in \langle \omega \rangle$, $x \in (0, \infty)$, be a family of random variables independent of the S.M.P. $\varkappa_t(s)$, independent in the aggregate, whose distribution does not depend on the index k . Denote

$$\nu_t(j) = \min\{n : \varepsilon_n \in \overline{\langle \omega \rangle} \mid \varepsilon_0 = j\}, \quad j \in \langle \omega \rangle.$$

Introduce an additive functional of the trajectory of the S.M.P. of the form

$$\xi_t(j) = \sum_{k=0}^{\nu_t(j)-1} f_t^{(k)}(\varepsilon_k, \tau_t(\varepsilon_k)), \quad \text{if } \varepsilon_0 = j.$$

We shall study the limiting distributions of $\xi_t(j)$ under the corresponding normalization, as $t \rightarrow \infty$. Denote by $\tau_t(l, j)$ a random variable with distribution function $p_t(l, j)^{-1} F_t(l, j, u)$, $l, j \in \langle \omega \rangle$, and let

$$\gamma_t(l, j) = f_t^{(1)}(l, \tau_t(l, j)), \quad \psi_t(l, j, \lambda) = M \exp\{i\lambda\gamma_t(l, j)\}.$$

Put

$$g_t = \sum_{k \in \langle \omega \rangle} q_t(k, \langle \omega \rangle) \sum_{i \in \langle \omega \rangle} p_t(k, i),$$

where $q_t(k, \langle \omega \rangle)$, $k \in \langle \omega \rangle$, is the stationary distribution for the chain with matrix $\tilde{P}(t, \langle \omega \rangle) = \|\tilde{p}_t(k, j, \langle \omega \rangle)\|$, $k, j \in \langle \omega \rangle$, and

$$\tilde{p}_t(k, j, \langle \omega \rangle) = p_t(k, j) \left(1 - \sum_{i \in \langle \omega \rangle} p_t(k, i) \right)^{-1}.$$

Here $p_t(k, j) = F_t(k, j, \infty)$, $k, j = 0, \dots, r$, are the transition probabilities for the embedded Markov chain. Suppose that there exists $\beta = \beta(g_t) \rightarrow 0$ such that, for any $l, j \in \langle \omega \rangle$,

$$\psi_t(l, j, \beta\lambda) = 1 + a_{lj}(\lambda) g_t(q_t(l, \langle \omega \rangle) p_t(l, j))^{-1} (1 + o(1)), \quad (1)$$

where $|a_{ij}(\lambda)| \geq 0$. Note that this condition is equivalent to the condition that the quantities $\gamma_t(l, j)$ are attracted to certain infinitely divisible laws. Denote $\varphi_t(j, \lambda) = M \exp\{i\lambda\xi_t(j)\}$.

Theorem 1. *If the set $\langle \omega \rangle$ forms a group and (1) is satisfied, then, independently of the initial state $j \in \langle \omega \rangle$, as $t \rightarrow \infty$*

$$\varphi_t(j, \beta\lambda) \rightarrow \frac{1}{1 - a(\lambda)};$$

here

$$a(\lambda) = \sum_{i, j \in \langle \omega \rangle} a_{ij}(\lambda)$$

is the logarithm of the characteristic function of a certain infinitely divisible law.

Remark 1. In the case when $a_{ij}(\lambda) = c_{ij}\lambda^\alpha$, $i, j \in \langle \omega \rangle$ (c_{ij} are certain constants), the theorem has a simple probabilistic meaning:

$$\beta \xi_t(j) \xrightarrow{\text{sl}} \xi \eta^{1/\alpha *},$$

where ξ is a stable distribution with exponent α , $0 < \alpha \leq 2$, η is an exponential distribution, and ξ and η are independent.

Remark 2. If, for the quantities $\gamma_t(k, j)M_t^{-1}$, $k, j \in \langle \omega \rangle$, condition (1) is satisfied, where $a_{kj}(\lambda) = i\lambda c_{kj}$, and, in addition, the quantities $\gamma_t(k, j)$ have an n -th moment ($n \geq 1$), with

$$M_t^{-l} g_t^{l-1} q_t(k, \langle \omega \rangle) p_t(k, j) M \gamma_t(k, j)^l = o(1),$$

$$1 < l \leq n, \quad k, j \in \langle \omega \rangle,$$

then under the conditions of Theorem 1

$$M_t^{-1} g_t \xi_t(j) \xrightarrow{\text{sl}} \eta$$

and all moments up to order n , inclusive, converge. Here

$$M_t = \sum_{i \in \langle \omega \rangle} q_t(i, \langle \omega \rangle) M f_t^{(1)}(i, \tau_t(i)),$$

η is exponential and $M\eta = 1$. In particular, if we set $f_t^{(k)}(i, x) = 1$ or $f_t^{(k)}(i, x) = x$, $i \in \langle \omega \rangle$, we obtain that

$$g_t \nu_t(j) \xrightarrow{\text{sl}} \eta, \quad m_t^{-1} g_t \Omega_t(i) \xrightarrow{\text{sl}} \eta,$$

where $\Omega_t(j)$ is the time spent by the s.m.p. in the group $\langle \omega \rangle$ before exit, under the condition that $\varepsilon_0 = j \in \langle \omega \rangle$, and

$$m_t = \sum_{i \in \langle \omega \rangle} q_t(i, \langle \omega \rangle) M \tau_t(i).$$

This agrees with the results of ^(3, 4).

Let us now assume that the quantities $\gamma_t(k, j)$ have a first moment and that there exists $\beta = \beta(g_t)$ such that, for all $k, j \in \langle \omega \rangle$,

$$\psi_t(k, j, \beta \lambda) = 1 + i\lambda m_t(k, j)\beta + a_{kj}(\lambda) g_t \frac{1 + o(1)}{q_t(k, \langle \omega \rangle) p_t(k, j)}. \quad (2)$$

Theorem 2. *If the set $\langle \omega \rangle$ forms a group and (2) is satisfied, then, independently of the initial state $l \in \langle \omega \rangle$, the distribution of the random variable*

$$\alpha_t(\xi_t(l) - M_t \nu_t(l))$$

converges weakly to the distribution with characteristic function

$$\frac{1}{1 - c_1 a(\lambda) + c_2 \lambda^2},$$

* In the sense of weak convergence of distribution functions.

where $\alpha_t = \min\{\sqrt{g_t} M_t^{-1}, \beta\}$; $a(\lambda)$ is defined as in Theorem 1, and $c_1 \geq 0$ and $c_2 \geq 0$ are certain constants, with, if

$$\beta^2 M_t^2 g_t^{-1} \rightarrow \mu,$$

then for $\mu = \infty$, $c_1 = 0$, and for $\mu = 0$, $c_2 = 0$.

In conclusion we indicate an effective algorithm which, in a finite number of steps, not exceeding $r - 1$, makes it possible to check whether the given set $\langle \omega \rangle$ will form a group.

Call a transition from i to j “proper” if $P_t(i, j) \neq 0$. We perform the following operation with our embedded chain: at the first step we carry out all “proper” transitions. We obtain a certain directed graph. It is obvious that connected subsets from which there are no transitions form groups. Replace each such group $\langle d \rangle$ by one state with transition probabilities

$$p_t(\langle d \rangle, j) = \left(\sum_{k \in \langle d \rangle} q_t(k, \langle d \rangle) \sum_{l \in \langle d \rangle} p_t(k, l) \right)^{-1} \sum_{i \in \langle d \rangle} q_t(i, \langle d \rangle) p_t(i, j);$$

$$p_t(j, \langle d \rangle) = \sum_{k \in \langle d \rangle} p_t(j, k); \quad j \in \overline{\langle d \rangle}.$$

Thus, after the first step we obtain a new chain with a smaller number of states. With it we perform the same operation. Then it is clear that $\langle \omega \rangle$ forms a group if and only if all of $\langle \omega \rangle$ can be reduced to a single state.

It is of interest to note that the types of distributions found were indicated earlier in ^(5, 6) and, through the construction of $\xi_t(j)$, are naturally connected with the problem considered there.

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