

ON THE STABILITY OF ITERATIVE ALGORITHMS FOR SOLVING NONSELFADJOINT DIFFERENCE EQUATIONS

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Abstract

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MATHEMATICS

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ON THE STABILITY OF ITERATIVE ALGORITHMS FOR SOLVING NONSELFADJOINT DIFFERENCE EQUATIONS

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Let

$$u = R_N u + f_N \quad (1)$$

be a family of linear equations ("difference equations") for the unknown element u of some N -dimensional linear normed space U_N , depending on a natural parameter N . We shall consider the iterative process

$$u^{m+1} = R_N u^m + f_N, \quad m = 0, 1, \dots, \quad (2)$$

for computing the solution u . We shall assume that all eigenvalues $\lambda_k = \lambda_k(N)$ of the operator R_N are less than one in modulus,

$$|\lambda_k| < \rho_N < 1, \quad (3)$$

i.e., the well-known convergence criterion for the process (2) is satisfied; moreover,

$$\|u - u^m\| = o(\rho_N^m). \quad (4)$$

Suppose now that the computations (2) are carried out approximately, with a certain number $p = q + \alpha$ of decimal digits, i.e., according to the formula

$$\tilde{u}^{m+1} = R_N \tilde{u}^m + f_N + 10^{-p} \|\tilde{u}^m\| \delta_m, \quad (5)$$

where $\delta_m \in U_N$, $\|\delta_m\| \leq 1$, are arbitrary.

Let us prescribe a natural number q and require that, for arbitrary δ_k , $\|\delta_k\| \leq 1$, $k = 0, 1, \dots$, the inequality

$$\overline{\lim}_{m \rightarrow \infty} \|u - \tilde{u}^m\| \leq 10^{-q} \|u\| \quad (6)$$

hold.

Inequality (6) ensures the possibility of computing the solution u by formulas (5) with an error not exceeding one unit of the q -th decimal digit (in the sense of the norm in U_N).

Lemma. *For requirement (6) to hold, it is necessary that the number α of “guard digits” in formula (5) satisfy the inequality*

$$(1 - 10^{-q})\varphi \leq 10^\alpha,$$

and it is sufficient that α satisfy the inequality

$$(1 + 10^{-q})\varphi \leq 10^\alpha,$$

where

$$\varphi = \lim_{m \rightarrow \infty} \max_{\|\delta_k\|=1} \left\| \sum_{k=0}^m R_N^{m-k} \delta_k \right\|.$$

We note that the existence of $\varphi = \varphi(N)$ follows from condition (3). In what follows we shall understand by $\alpha = \alpha(N)$ the least integer ensuring the fulfillment of requirement (6). It is clear from the lemma that such a number exists, is nonnegative, and depends only insignificantly on q , or does not depend on it at all.

In accordance with the concept of stability of a computational process outlined in § 7 of the book ⁽¹⁾, we give the following

Definition. A convergent iterative algorithm (2) will be called **stable** if there exists a constant C , independent of N , for which the inequality

$$\alpha(N) < C; \quad (7)$$

holds; a convergent iterative algorithm will be called **weakly stable** if there exists a constant C , independent of N , for which the inequality

$$\alpha(N) < C \ln N, \quad (8)$$

holds, but stability does not occur. Finally, a convergent iterative algorithm will be called **unstable** if it is neither stable nor weakly stable.

Example. Let us write the equation

$$\begin{aligned} -2u_n + u_{n+1} + f_n &= 0, & n = 0, 1, \dots, N-1, \\ u_N &= 0 \end{aligned} \quad (9)$$

in the form

$$\begin{aligned} u_n &= (1-2r)u_n + ru_{n+1} + rf_n, & n = 0, 1, \dots, N-1, \\ u_N &= 0, \end{aligned} \quad (10)$$

where $r > 0$ is a parameter. The iterative algorithm (2) for equation (10) takes the form

$$\begin{aligned} u_n^{m+1} &= (1-2r)u_n^m + ru_{n+1}^m + rf_n, & n = 0, 1, \dots, N-1, \\ u_N^{m+1} &= 0, \end{aligned} \quad (11)$$

so that the operator R_N , $v = R_N u$, is written by the formulas

$$\begin{aligned} v_n &= (1-2r)u_n + ru_{n+1}, \\ v_N &= 0. \end{aligned}$$

The operator R_N has, as is easy to see, only two eigenvalues $\lambda_1(N) = 1-2r$ and $\lambda_2(N) = 0$.

Inequality (3) is satisfied and the iterative algorithm (11) converges for $r < 1$. Let us take as the norm $\|u\| = \max |u_n|$. We show that for $r < 2/3$ it is stable, and for $r > 2/3$ unstable. Indeed, if $r < 2/3$, then

$$\max_n |v_n| \leq \max(|1-3r|, |1-r|) \max_n |u_n|,$$

so that $\|R_N\| \leq \max(|1-3r|, |1-r|) = \rho < 1$. Therefore $\varphi(N) \leq 1/(1-\rho)$, and by virtue of the lemma the estimate (7) holds with the constant $C = -2 \ln(1-\rho)$. Now let $r > 2/3$. Put in (11) $f_n = 0$, $u_n^0 = (-1)^n$, $n = 0, 1, \dots, N-1$. It is easy to see that in this case $u_n^m = (1-3r)^m (-1)^n$, $n = 0, 1, \dots, N-m$. Hence it follows that $\|R_N^m\| \geq \rho^m$, $m = 1, 2, \dots, N-1$, where $\rho = |1-3r| > 1$. Therefore $\varphi(N) \geq \rho^N$, and by virtue of the lemma $\alpha \approx N \lg \rho$, which proves instability. It can be shown that for $r = 2/3$ the iterative algorithm (11) is weakly stable. We note that the example considered is, in its construction, reminiscent of the example of S. K. Godunov ⁽²⁾, Ch. VI, constructed for another purpose.

Thus, the spectral convergence criterion (3) for an iterative algorithm does not determine its stability. The spectral criterion and the features of stability are

formulated not in terms of the location of the spectra of each of the operators R_N , but in terms of the location of the spectrum ⁽²⁾ and the spectral kernels ⁽³⁾ of the family of operators $\{R_N\}$. Namely, under the assumption that the family of operators R_N is uniformly bounded, $\|R_N\| < C$, the following assertions are valid:

Lemma. In order that, for all sufficiently large values of N , the iterative algorithm (2) be convergent, it is sufficient that the radius ρ of some kernel of the spectrum of the family of operators $\{R_N\}$ be strictly less than one.

Stability criterion. For the stability of the iterative algorithm (2), it is necessary and sufficient that the spectrum of the family of operators $\{R_N\}$ lie strictly inside the unit circle.

Theorem. In order that the iterative algorithm (2) converge and be stable or weakly stable, it is sufficient that the radius ρ of the arithmetic kernel of the spectrum of the family of operators $\{R_N\}$ be strictly less than one; for instability of the convergent iterative algorithm (2), it is sufficient that the radius ρ of the arithmetic kernel of the spectrum of the family of operators $\{R_N\}$ be strictly greater than one.

In ⁽³⁾ it is shown that the arithmetic kernel of the spectrum of a family of operators $\{R_N\}$ does not depend on the choice of norms from the class of norms natural for difference equations. Hence it follows, in particular, that if the operators R_N are uniformly in N contractions, $\|R_N\| < C < 1$, so that the spectrum, and therefore also the arithmetic kernel of the spectrum of the family of operators $\{R_N\}$, lie in the circle $|\lambda| < C$, then the iterative algorithm (2) is stable and remains stable (strongly or weakly) in any other natural ⁽³⁾ norm, in which the operators R_N may cease to be contractions.

In the example considered above, the spectrum of the family of operators $\{R_N\}$ consists of the circle $|\lambda - (1 - 2r)| \leq r$ and the point $\lambda = 0$, and coincides with its arithmetic kernel. The assertion on the stability of algorithm (11) for $r < 2/3$ and on instability for $r > 2/3$ can therefore also be made by means of spectral criteria.

For computing the solution of a (non-self-adjoint) equation of the form

$$A_N u + f_N = 0 \quad (12)$$

one may try to construct an iterative algorithm in the form

$$B_N u^{m+1} = B_N u^m + (A_N u^m + f_N). \quad (13)$$

Here the operator B_N must be chosen so that it is easy to invert numerically and so that the spectrum of the family of operators $\{R_N\}$, $R_N = E_N + B_N^{-1} A_N$, has the smallest possible radius ρ , $\rho < 1$. By virtue of the estimate $\|R_N^m\| \leq C(\varepsilon)(\rho + \varepsilon)^m$, where $\varepsilon > 0$ is arbitrary and $C(\varepsilon)$ does not depend on N , proved

in ⁽³⁾, this will ensure rapid convergence; and, by virtue of the stability criterion formulated above, the stability of the iterative algorithm (13).

The choice of the operator B_N on the basis of spectral considerations in the case of self-adjoint and nearly self-adjoint difference equations (12) was used by Douglas and Rachford ⁽⁴⁾, E. G. Dyakonov (see ⁽⁵⁾ and the bibliography given there), and others.

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