

A THEOREM ON EXTERNALLY STABLE SETS

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Abstract

Full Text

UDC 519

MATHEMATICS

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A THEOREM ON EXTERNALLY STABLE SETS

(Presented by Academician Yu. V. Linnik on 23 X 1969)

Consider a cooperative game Γ with the set of players $I = \{1, 2, \dots, n\}$ and characteristic function $v(S)$, given in 0-1-reduced form, i.e.

$$v(i) = 0, \quad i = 1, 2, \dots, n; \quad v(I) = 1.$$

Then the set of imputations is

$$A = \left\{ x = (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^n x_i = 1 \right\}.$$

As is known, a solution is a set of imputations $V \subset A$ possessing internal and external stability.

The core is a considerably simpler object than a solution; it has the form

$$U = \left\{ x \in A : \sum_{i \in S} x_i \geq v(S), \quad \forall S \subset I \right\}.$$

If all $v(S) = 0$, $S \subset I$, $S \neq I$, then $A = U$ and U is a solution; known are also nontrivial sufficient conditions which $v(S)$ must satisfy for the core to be a solution (see ^(1,2)).

The question arose whether it is possible to construct some transition from the core to a solution. It turns out that if, in a three-person game, one continuously increases $v(S)$, $|S| = 2$, up to some $v'(S)$ at which the core is still nonempty, then the latter, together with the traces of the intersections of its boundaries during this process (see the precise definitions below), gives a solution of the game. In the general case one can prove only the external stability of such a set.

We now formulate the problem mathematically. Consider a family of games Γ_t , $0 \leq t \leq 1$, with the set of players $I = \{1, 2, \dots, n\}$, the same for all games, and let

$$v_t(i) \equiv 0, \quad 0 \leq t \leq 1,$$

$$v_t(I) \equiv 1, \quad 0 \leq t \leq 1,$$

and let $v_t(S) = f_S(t)$, $2 \leq |S| \leq n-1$, be nondecreasing continuous functions on $0 \leq t \leq 1$.

Denote by $U(t)$ the core of the game Γ_t , by $V(t)$ the solution, and by $F(t)$ the following set:

$$F(t) = \bigcup_{0 \leq \tau < t} \bigcup_{S \neq T} \left\{ \begin{array}{l} x \in U(\tau), \\ \sum_{i \in S} x_i = f_S(\tau), \quad 2 \leq |S| \leq n-1, \\ \sum_{i \in T} x_i = f_T(\tau), \quad 2 \leq |T| \leq n-1. \end{array} \right.$$

Put $W(t) = F(t) \cup U(t)$. Note that the set of imputations A is the same for all games Γ_t .

Theorem. *If the family of games Γ_t , $0 \leq t \leq 1$, is such that: 1) $U(0) = V(0)$; 2) $U(1) \neq \Lambda$, then the set $W(t)$ is externally stable in the game Γ_t for every $0 \leq t \leq 1$.*

Proof. Fix some $t : 0 < t \leq 1$ (for $t = 0$ the theorem is true by condition 1)). The external stability of $W(t)$ means that, whatever $z \in A - W(t)$ may be, there exists an $x \in W(t)$ such that $x > z$.

Consider two possibilities.

I. $z \notin U(\tau)$, $0 \leq \tau \leq t$. Since $U(0) = V(0)$ and $z \notin U(0)$, there exist $y \in U(0)$ and an essential S_0 such that

$$y \underset{S_0}{>} z. \quad (1)$$

Consider the sets

$$h_{S_0}(\tau) = U(\tau) \cap \left\{ x : \sum_{i \in S_0} x_i = f_{S_0}(\tau) \right\},$$

$$H_{S_0}(t) = \bigcup_{0 \leq \tau \leq t} h_{S_0}(\tau)$$

and the convex open cone

$$K_{S_0}(z) = \{x \in A : x_i > z_i, \forall i \in S_0\}.$$

In view of condition (1), $K_{S_0}(z) \cap h_{S_0}(0) \neq \Lambda$.

- 1) If $H_{S_0}(t) \in F(t)$, then $y \in W(t)$, and $y > z$; there remains the case:
- 2) $H_{S_0}(t) \notin F(t)$, i.e., there exists an $x \in H_{S_0}(t)$ such that

$$\sum_{i \in S} x_i > v(S),$$

$S \subset I$, $S \neq S_0$; this means that $H_{S_0}(t)$ is a closed domain of dimension $(n-1)$ in Euclidean space E_{n-1} . Since $K_{S_0}(z) \cap h_{S_0}(0) \neq \Lambda$, the open cone $K_{S_0}(z)$ intersects the set $H_{S_0}(t)$ at some point ω lying on the boundary of $H_{S_0}(t)$, but not belonging to $h_{S_0}(0)$. If, in addition:

a) $\omega_i > 0$, $i = 1, 2, \dots, n$, then $\omega \in F(t)$, hence $\omega \in W(t)$, and $\omega > z$; there remains the case b) $\omega_{i_0} = 0$ and $\omega \notin F(t)$; necessarily then $i_0 \notin S_0$; let

$$\tau_0 : \sum_{i \in S_0} \omega_i = f_{S_0}(\tau_0) > f_{S_0}(0),$$

then

$$\sum_{i \in T} \omega_i > f_T(\tau_0), \quad T \neq S_0. \quad (2)$$

Consider the allocation $a(\mathcal{E}) = (a_1(\mathcal{E}), \dots, a_n(\mathcal{E}))$:

$$a_i(\mathcal{E}) = \omega_i + \frac{\mathcal{E}}{|S_0|}, \quad i \in S_0,$$

$$a_j(\mathcal{E}) = \omega_j - \mathcal{E}_j, \quad j \notin S_0,$$

$$\mathcal{E} \geq 0, \quad \mathcal{E}_j \leq \omega_j, \quad \sum_{j \notin S_0} \mathcal{E}_j = \mathcal{E}.$$

It is easy to show that, by continuously increasing $\mathcal{E}$, we finally obtain such an $\bar{\mathcal{E}}$ that either $a(\bar{\mathcal{E}}) \in U(t)$, or at least two conditions from (2) are violated, i.e. $a(\bar{\mathcal{E}}) \in F(t)$. Hence, also in case b), there exists $a(\bar{\mathcal{E}}) > z$, $a(\bar{\mathcal{E}}) \in W(t)$.

II. $z \in U(\tau_0)$, $0 \leq \tau_0 < t$, i.e. $z \in U(0)$. Consider the sets $H_S(t)$ for all $S : 1 < |S| < n$ (they may be both empty and repeating). It is clear that

$$\bigcup_S H_S(t) \cup U(t) = U(0);$$

since $z \notin U(t)$, there exists a nonempty $H_{S_0}(t) : z \in H_{S_0}(t)$. Consider the cone $K_{S_0}(z)$; since

$$\sum_{i \in S_0} z_i \geq v(S_0)$$

and $K_{S_0}(z)$ is open, $K_{S_0}(z) \cap h_{S_0}(0) = \Lambda$. Since $z \notin F(t)$, $H_{S_0}(t)$ in the space E_{n-1} is a closed set of the same dimension, the point z is an interior point of $H_{S_0}(t)$; hence the cone $K_{S_0}(z)$ intersects $H_{S_0}(t)$ at a point $y \notin h_{S_0}(0)$. Further, as in case I, 2), considering two possibilities, we complete the proof.

Example. Consider a market with one seller, player 1, and buyers, players $2, 3, \dots, n$. Then $v(S) = 0$, $1 \notin S$, and let $v(1, 2, \dots$

$$\dots, n) = 1.$$

If $v(S) \leq 1/(n - |S| + 1)$, $1 \in S$, then, by the theorem of T. E. Kulakovskaya, the game has a solution coinciding with the core.

Consider the transformation

$$f_{S_0}(t) = t, \quad S_0 = (2, 3, \dots, n); \quad f_S(t) \equiv v(S), \quad S \neq S_0;$$

it is obvious that Γ_t , for any $0 \leq t \leq 1$, has a nonempty core.

Consider the set $W(t)$, $0 \leq t \leq 1$; it is externally stable in Γ_t by the theorem. Its internal stability can be violated because of domination by the coalition S_0 . But for any $x = (x_1, \dots, x_n)$:

$$\sum_{i=2}^n x_i = t,$$

$$\sum_{i \in S} x_i \geq v(S),$$

if $y = (y_1, \dots, y_n)$ is such that $y_i < x_i$, $i = 2, \dots, n$, then

$$\sum_{i \in S} y_i > v(S), \quad 1 \in S,$$

i.e. $\text{dom } h_{S_0}(t)$ lies inside the core $U(0)$, that is, $W(t)$ is internally stable.

Thus, a solution has been described for any game:

$$v(S) \leq 1/(n - |S| + 1), \quad 1 \in S; \quad v(S) = 0, \quad S \subset \{2, \dots, n\};$$

$$v(2, 3, \dots, n) \leq 1.$$

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CITED LITERATURE

1. O. N. Bondareva, *Problems of Cybernetics*, no. 10, 1963.

Note: Figure translations are in progress. See original paper for figures.

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