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1970

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Abstract

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UDC 519.212.1

MATHEMATICS

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DEFINITION OF COMPLEX MARKOV CHAINS AND DERIVATION OF THE SCHRÖDINGER EQUATION

(Presented by Academician A. N. Tikhonov on 26 VIII 1969)

1. Let $\Omega = \{A_1, A_2, \dots, A_n\}$ be a finite set with the σ -algebra $\mathfrak{A}$ of all its subsets, on which a probability $P(A)$, $A \in \mathfrak{A}$, is defined. We shall call the elements of the algebra $\mathfrak{A}$ true events, and the elements $A_1, A_2, \dots, A_n$ elementary true events. Let A_i itself be subdivided into k elements

$$A_i = \{B_1^i, B_2^i, \dots, B_k^i\},$$

which we shall call elementary virtual events. Their totality (for $i = 1, 2, \dots, n$) will be called the set of elementary events, and the elements of the σ -algebra $\mathfrak{B}^i$ of subsets of A_i will be called virtual events.

Let $Z(B)$ be a certain complex function on the σ -algebra $\mathfrak{B}$ generated by the set of nk elementary virtual events, additive on each of the sets A_i and such that

$$|Z(A)|^2 = P(A), \quad A \in \mathfrak{A}. \quad (1)$$

We shall call the elements of the σ -algebra $\mathfrak{B}$ simply events, and the function $Z(B)$, $B \in \mathfrak{B}$, the amplitude of the event B .

Definition 1. The **conditional amplitude** $Z(B_1/B_2)$ of the event B_1 upon the occurrence of the event B_2 is defined as the ratio

$$Z(B_1/B_2) = \frac{Z(B_1 \cdot B_2)}{Z(B_2)}.$$

2. We shall consider a finite system $\mathcal{K}$ of states $a_1, \dots, a_k$, which replace one another at finite time intervals $T_0, T_1, \dots, T_N$. As elementary true events A_{ij} we take all possible pairs of points $\{(T_0, a_i), (T_N, a_j)\}^*$ in the plane T, a . We subdivide this event into elementary virtual events that are sequences of N points

$$\{(T_0, a_i), (T_1, a_{i_1}), \dots, (T_{N-1}, a_{i_{N-1}}), (T_N, a_j)\},$$

in which the initial and final points are fixed. In other words, the elementary virtual events are all possible “paths” of the system $\mathcal{K}$ from the point (T_0, a_i) to the point (T_N, a_j) . The set of all paths joining all initial and final points is the set of elementary events.

Definition 2. Suppose that on the σ -algebra of paths an amplitude $Z(B)$ is given. We shall say that it **defines a complex Markov chain** (a **KM-chain**) if the conditional amplitude of the fact that the system is in the state^{**} a_ν at the time T_{i+1} , under the condition that at the moments $T_i, T_{i-1}, \dots, T_0$ it was respectively in the states $a_\mu, a_{\mu_1}, \dots, a_{\mu_{i-1}}$, is equal to the conditional amplitude of the fact that the system is in the state a_ν at the time T_{i+1} , under the condition that at the time T_i it was in the state

* This is connected with the physical fact that we can perform measurements only at the initial and final moments.

** The event (T_i, a_ν) , consisting in the fact that the system at the time T_i is in the state a_ν , is the subset of all paths passing through the point (T_i, a_ν) . The true event $A_{\mu\nu}$ is equal to the intersection of the event (T_0, a_μ) and the event (T_N, a_ν) .

a_μ . We shall call this amplitude the **transition amplitude** from a_μ to a_ν in one step and denote it by $z_{\mu\nu}^{i,i+1}$.

Definition 3. A KM -chain is called **homogeneous** if its one-step transition amplitudes do not depend on T_i and N .

Let us put $N = 1$ in a homogeneous KM -chain. In this case there are no virtual events. Therefore we shall regard such a situation as purely classical, and shall obtain the transition probabilities $P_{\mu\nu}^{0,1} = |z_{\mu\nu}^{0,1}|^2$ from considerations of classical probability theory. The transition probabilities for a homogeneous KM -chain with $N = 1$ will be called the **classical transition probabilities** of the KM -chain (for arbitrary N).

Let us pass to continuous KM -processes. Let the system $\mathcal{K}$ consist of the points $x_\nu = \varepsilon\nu$, $|x_\nu| \leq l(\varepsilon)$, where $l(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, and suppose, moreover, that $T_{i+1} - T_i \rightarrow 0$ as $\varepsilon \rightarrow 0$, while $N \rightarrow \infty$ in such a way that $T_N \rightarrow t$. We shall weaken relation (1), putting

$$|Z(A)|^2 = P(A) + o(\varepsilon), \quad A \in \mathfrak{A}. \quad (2)$$

In this case (with the other conditions preserved) we shall say that the amplitude $Z(B)$ **defines** a homogeneous ε - KM -chain (cf. (1)).

We shall say that a homogeneous ε - KM -chain **defines a continuous KM -process** if there exists a function $G(x, \xi, t)$ such that: 1) $|G(\mu\varepsilon, \nu\varepsilon, T_i) - \varepsilon^{-1}Z((T_i, x_\nu)/(T_0, x_\mu))| \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in μ, ν, i ; 2)

the convolutions, in the second argument, of the first derivatives of $G(x, \xi, t)$ at $t = 0$ with functions of the class S (the fundamental functions) belong to S ; 3) at those points x at which $G(x, \xi, t)$ is less than some constant, it is infinitely differentiable. The function $G(x, \xi, t)$ will be called the **Green' s function** of the limiting KM -process.

Now consider nonrelativistic particles in an electromagnetic field, emitted from the point ξ with an equiprobable distribution of velocities.

The velocity of a particle which, in the time t , has arrived at the point x_i , is equal to $v_i = (x_i - \xi)/t + O(1)$ as $t \rightarrow 0$. The probability of arriving in the interval $[x_1, x_2]$ is, obviously, proportional to $v_2 - v_1 = (x_2 - x_1)/t + O(1)$. Therefore, for the ε - KM -chain corresponding to the motion of a nonrelativistic particle in an electromagnetic field, according to what has been said above, the relation

$$\sum_{j_1 \leq j \leq j_2} P_{ij}^{0,1} = (x_{j_2} - x_{j_1})/(T_1 - T_0) + O(1), \quad (3)$$

must hold, where $\beta P_{ij}^{0,1}$ are the classical transition probabilities of the KM -chain, β is a normalizing constant.

Theorem 1. In order that $G(x, \xi, t)$ be the Green' s function of a KM -process that is limiting for some ε - KM -chain whose classical transition probabilities satisfy condition* (3), it is necessary and sufficient that it satisfy the equation (Schrödinger)

$$i \partial G / \partial t = \{[i\alpha \partial / \partial x + A(x)]^2 + V(x)\} G,$$

where $A(x)$, $V(x)$ are certain smooth functions, α is a constant.

The proof of sufficiency follows easily from the asymptotics, as $t \rightarrow 0$, of the Green' s function of the Schrödinger equation. We outline the proof of necessity. It is based on a number of lemmas.

Lemma 1. The conditional amplitude $Z((T_{i+j}, a_\nu)/(T_i, a_\mu))$ of a KM -chain is equal to the matrix element $a_{\mu\nu}$ of the product of matrices

$$\text{Mat } a_{\mu\nu} = \text{Mat } z_{\mu\nu}^{i,i+1} \text{Mat } z_{\mu\nu}^{i+1,i+2} \dots \text{Mat } z_{\mu\nu}^{i+j-1,i+j}.$$

The proof is analogous to the proof of the corresponding assertion for Markov chains (2). It follows from it that

Lemma 2. The operator

$$U_t \psi(x) = \int G(x, \xi, t) \psi(\xi) d\xi,$$

where $G(x, \xi, t) -$

* That is, for a nonrelativistic particle in an electromagnetic field.

function, the Green function of the limiting CM-process, is a unitary one-parameter group.

Lemma 3. For the Green function of the limiting CM-process satisfying the condition of the theorem, the following asymptotic relation holds:

$$|G(x, \xi, t)|^2 = \text{const} / t + O(1) \quad (4)$$

as $t \rightarrow 0$.

Let $\hat{H}$ be the generating self-adjoint operator of the group U_t . The Green function $G(x, \xi, t)$ obviously satisfies the equation

$$i \partial G / \partial t = \hat{H} G, \quad G|_{t=0} = \delta(x - \xi). \quad (5)$$

From relation (4) one may conclude that $G(x, \xi, t)$ has a singularity only at $t = 0$. Consequently, the approximate solution of problem (5), obtained up to smooth functions, coincides with the asymptotics of the solution as $t \rightarrow 0$ (cf. (3)). The operator $\hat{H}$ can be represented as the pseudodifferential operator

$$\hat{H}\psi(x) = i \lim_{t \rightarrow 0} \int (\partial G / \partial t) \psi(\xi) d\xi = \int H(p, x) \psi(\xi) \exp[ip(x - \xi)] d\xi, \quad \psi(x) \in S \quad (6)$$

with a certain complete symbol $H(p, x)$, which, by virtue of property 2) for $G(x, \xi, t)$, turns out to be sufficiently smooth and to grow more slowly than any exponential.

The construction of the solution of problem (5), where $\hat{H}$ is a pseudodifferential operator, up to smooth functions (for small t) is carried out by the method of the work (5). It turns out that the following is valid.

Lemma 4. The asymptotics as $t \rightarrow 0$ of the solution of problem (5) satisfying condition (4) has the form

$$|\partial^2 S / \partial x \partial \xi|^{-1/2} \{e^{iS(x, \xi, t)} + O(t)\},$$

where $S(x, \xi, t)$ satisfies the Hamilton-Jacobi equation

$$\partial S / \partial t + H(\partial S / \partial x, x) = 0.$$

Hence, and from (4), it is concluded that

$$S(x, \xi, t) = a(x - \xi)^2/t + O(1).$$

Therefore

$$\int G(x, \xi, t)(x - \xi)^3 F(\xi) d\xi = o(t) \quad (7)$$

for any finite $F(\xi)$.

From equality (6) it follows that the limits* exist

$$\begin{aligned} \lim_{t \rightarrow 0} t^{-1} \left[\int G(x, \xi, t) e(\xi) d\xi - e(x) \right] &= V_N(x), \\ \lim_{t \rightarrow 0} t^{-1} \int G(x, \xi, t)(x - \xi) e(\xi) d\xi &= A_N(x), \\ \lim_{t \rightarrow 0} t^{-1} \int G(x, \xi, t)(x - \xi)^2 e(\xi) d\xi &= 2B_N(x), \end{aligned} \quad (8)$$

where $e(\xi)$ is a finite function equal to 1 for $\xi \in (-N, N)$.

Let $\psi_0(x)$ be finite, $\text{supp } \psi_0 \in (-N, N)$. Since

$$\begin{aligned} \psi_0(\xi) &= \psi_0(x) e(\xi) + (x - \xi) \psi'_0(x) e(\xi) + \\ &+ \frac{1}{2} (x - \xi)^2 \psi''_0(x) e(\xi) + (x - \xi)^3 F(x, \xi) e(\xi), \end{aligned}$$

then, by virtue of (7) and (8),

$$\begin{aligned} \int G(x, \xi, t) \psi_0(\xi) d\xi - \psi_0(x) &= \\ &= t [V_N(x) \psi_0(x) + A_N(x) \psi'_0(x) + B_N(x) \psi''_0(x)] + o(t). \end{aligned}$$

* Cf. A. N. Kolmogorov' s derivation of the equation for a limiting Markov process.

Consequently,

$$\hat{H} \psi_0 = B_N \psi''_0 + A_N \psi'_0 + V_N \psi_0. \quad (9)$$

Hence, again taking into account (4) and Lemma 4, and then carrying out the closure procedure ($N \rightarrow \infty$), we arrive at the assertion of the theorem, since, by virtue of (9), for $N_1 < N_2$ and $x \in (-N_1, N_1)$ the equalities $A_{N_1}(x) = A_N(x)$, $B_{N_1}(x) = B_N(x)$, $V_{N_1}(x) = V_N(x)$ hold.

The generalization to the inhomogeneous (time-dependent) multidimensional case can be carried out without substantial changes.

In conclusion, the author expresses his gratitude to V. M. Alekseev for a valuable discussion.

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Received
23 VIII 1969

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