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# ASYMPTOTIC BEHAVIOR OF QUASIANALYTIC FUNCTIONS

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## Abstract

## Full Text

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MATHEMATICS

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# ASYMPTOTIC BEHAVIOR OF QUASIANALYTIC FUNCTIONS

(Presented by Academician V. I. Smirnov on 13 VII 1970)

1°. Let  $C\{M_n\}$  be a quasianalytic class of functions  $f(x)$  defined on the entire real axis,  $\|f\| = \sup_n M_n^{-1} \max_x |f^{(n)}(x)| < \infty$ . Denote by  $I_a(F)$  the subset of  $C\{M_n\}$  consisting of functions  $f(x)$  for which  $|f(x)| = O(F(x-a))$ . B. I. Korenblum <sup>(1)</sup> established the existence of such a function  $F$  that  $I_a(F)$  is a nonempty closed subspace of  $C\{M_n\}$  and  $\bigcap_a I_a(F) = \emptyset$ . This function determines the limiting rate of decrease in the class  $*C\{M_n\}$ .

In the present note it is shown that every function from the class  $C\{M_n\}$  is representable as the sum of a function from  $I_a(F)$  and a function holomorphic and belonging to the class  $**A$  in a certain narrowing infinite strip  $D_a(F)$ . The strip  $D_a(F)$  is characterized by the following property: if  $\Psi(z)$  maps  $D_a(F)$  conformally onto the right half-plane, then  $\Psi(x) = O(-\ln F(x))$ . This result makes it possible to establish an analogue of the well-known Ahlfors-Heins theorem <sup>(4)</sup> for quasianalytic functions and thereby to give a description of their asymptotic behavior along the real axis.

2°. Without loss of generality one may assume that the sequence  $\{M_n\}$  is logarithmically convex,  $\lim_{n \rightarrow \infty} (n^{-1} M_n)^{1/n} = \infty$ ,  $M_0 = 1$  <sup>(5)</sup>.

Simple computations show that, under these conditions, the function

$$\omega(x) = -\ln \left[ x^4 \min_{n \geq 0} x^{-n} M_n \right], \quad x \geq 1,$$

has the following properties:  $0 \leq \omega(x_1 + x_2) \leq \omega(x_1) + \omega(x_2)$ ,

$$\frac{x\omega'(x)}{\omega(x)} \nearrow 1 \quad \text{as } x \rightarrow \infty, \quad \int_1^\infty x^{-2} \omega(x) dx < \infty.$$

Put  $\omega(\varphi_n) = n$ ,

$$\alpha(x) = \prod_{n=1}^{\infty} (1 + x^2 \varphi_n^{-2})^{1/2}.$$

The following lemma establishes the connection between functions of the class  $C\{M_n\}$  and Fourier transforms of functions integrable with weight  $\alpha(x)$ .

**Lemma 1.** Let  $f(x) \in C\{M_n\}$ ,

$$g(x) = \int_{-\infty}^{\infty} f(\tau)(1 + \tau^2)^{-1} e^{ix\tau} d\tau.$$

\* An earlier weaker result was obtained by Hirschman <sup>(2)</sup>.

\*\* A function  $\varphi(z)$  belongs to the class  $A$  in the strip  $D_a(F)$  if the image  $\varphi$  under a conformal mapping of  $D_a(F)$  onto the right half-plane belongs to the class  $A$  (<sup>(3)</sup>, p. 289).

Then

$$\|g\|_{\alpha} = \int_{-\infty}^{\infty} |g(x)|\alpha(x) dx \leq K\|f\|,$$

where  $K$  does not depend on  $f$ .

Put

$$\xi(x) = \sum_{n=1}^{\infty} \frac{x^2}{\varphi_n(x^2 + \varphi_n^2)}, \quad \eta(\xi(x)) = x,$$

$$F(x) = \begin{cases} \exp \left[ -\frac{1}{10} \exp \left( \frac{\pi}{2} \int_1^x \frac{\eta(t)}{\omega(\eta(t))} dt \right) \right], & \text{for } x \geq 2, \\ F(2), & \text{for } x < 2, \end{cases}$$

and, for real  $a$ , denote by  $D_a(F)$  the domain in the complex plane  $z = \sigma + i\tau$  defined by the inequalities

$$|\tau| \leq \frac{\omega(\eta(\sigma - a))}{\eta(\sigma - a)}$$

for  $\sigma \geq a + 1$ , and  $|\tau| \leq \omega(\eta(1))/\eta(1)$  for  $\sigma < a + 1$ .

**Lemma 2.** If  $\Psi(z)$  maps  $D_1(F)$  conformally onto the right half-plane, then  $\Psi(x) = O(-\ln F(x))$ .

For functions integrable with weight  $\alpha(x)$ , the following holds.

**Theorem 1.** Let

$$\int_{-\infty}^{\infty} |g(x)|\alpha(x) dx < \infty, \quad \tilde{g}(x) = \int_{-\infty}^{\infty} g(t)e^{ixt} dt.$$

Then for every  $a \in (-\infty, \infty)$  there is a representation

$$\tilde{g}(x) = \tilde{g}_a^-(x) + \tilde{g}_a^+(x),$$

where  $\tilde{g}_a^+(x)$  is the restriction to the real axis of an entire function  $\tilde{g}_a^+(z)$  belonging to class  $A$  in the domain  $D_a(F)$ ,  $\tilde{g}_a^-(x) \in I_a(F)$ , and

$$\sup_x |\tilde{g}_a^-(x)| \leq c_g(a) \|g\|_\alpha, \quad \lim_{a \rightarrow -\infty} c_g(a) = 0.$$

**Remark 1.** From the results of Warschawski <sup>(6)</sup> it follows that, if a function  $\psi(z)$  belongs to class  $A$  in the strip  $D_a(F)$  and  $\psi(\sigma) \in I_{a-\varepsilon}(F)$ ,  $\varepsilon > 0$ , then  $\psi(z) \equiv 0$ . Therefore  $\tilde{g}_a^+(x)$  is the principal part of the function  $\tilde{g}(x)$ .

**Remark 2.** The strip  $D_a(F)$  in Theorem 1 cannot be substantially enlarged. Namely, it cannot be replaced by any domain containing every strip  $D_a(F)$  for sufficiently large  $\sigma$ .

The main role in the proof of Theorem 1 is played by

**Lemma 3.** Let

$$\Phi(x) = \prod_{n=1}^{\infty} \left(1 - \frac{ix}{\varphi_n}\right)^{-1} e^{ix/\varphi_n}, \quad \tilde{\Phi}(z) = \int_{-\infty}^{\infty} \Phi(x) e^{izx} dx, \quad z = \sigma + i\tau.$$

Then:

- a)  $\tilde{\Phi}(z)$  is an entire function, bounded in every half-plane  $\sigma \leq \sigma_0$ ;
- b)  $\tilde{\Phi}(z)$  belongs to class  $A$  in the strip  $D_0(F)$ ;
- c)  $\tilde{\Phi}(z) \in I_0(F)$ .

**Proof of Theorem 1.** Since  $\|g\|_\alpha < \infty$ , we have  $g(x) = \Phi(x)r(x)$ , where  $r(x) \in L^1(-\infty, \infty)$ . Therefore

$$\tilde{g}(x) = \int_{-\infty}^{\infty} \tilde{\Phi}(x-t)\tilde{r}(t) dt = \int_{-\infty}^a \tilde{\Phi}(x-t)\tilde{r}(t) dt + \int_a^{\infty} \tilde{\Phi}(x-t)\tilde{r}(t) dt = \tilde{g}_a^-(x) + \tilde{g}_a^+(x).$$

Using assertion b) of Lemma 3 and the inclusion  $D_p(F) \subset D_q(F)$  for  $p < q$ , we obtain that  $\tilde{g}_a^+(x)$  is the restriction to the real axis of a function  $\tilde{g}_a^+(z)$  belonging to class  $A$  in the strip  $D_a(F)$ . From assertion c) of Lem—

we and the inequality  $2F(x) \geq \int_x^{\infty} F(t) dt$ , we obtain that  $\tilde{g}_a^-(x) \in I_a(F)$ . The theorem is proved.

From Lemma 1 and Theorem 1 there follows the main

**Theorem 2.** Let  $f(x) \in C\{M_n\}$ . Then, for every  $a$ ,  $f(x)$  admits the representation

$$f(x) = f_a^+(x) + f_a^-(x),$$

where  $f_a^+(x)$  is the contraction to the real axis of a function  $f_a^+(z)$  belonging to the class  $A$  in the strip  $D_a(F)$ ,  $f_a^-(x) \in I_a(F)$ , and

$$\sup_x |f_a^-(x)| \leq c_f(a) \|f\|, \quad c_f(a) \leq 1, \quad \lim_{a \rightarrow -\infty} c_f(a) = 0.$$

**3°.** Let us apply the results obtained to the study of the asymptotic behavior of functions of the class  $C\{M_n\}$  as  $|x| \rightarrow \infty$ . To this end, we note that the asymptotic behavior of functions belonging to the class  $A$  in the domain  $D_a(F)$  is known. Namely, from the Ahlfors-Heins theorem <sup>(4)</sup> and the results of Warschawski <sup>(6)</sup> it follows that, for a function  $g(z) \neq 0$  belonging to the class  $A$  in  $D_a(F)$ , we have

$$\lim_{\sigma \rightarrow \infty}^* \frac{\ln |g(\sigma)|}{\ln F(\sigma - a)} = \gamma_g > -\infty,$$

where  $\lim^*$  means that, in tending to infinity,  $\sigma$  does not take values from a certain set  $E$ , for which

$$\int_E \frac{\eta(x)}{\omega(\eta(x))} dx < \infty.$$

Combining this result with Theorem 2, we obtain the following theorem.

**Theorem 3.** Let  $f(x) \in C\{M_n\}$ ,  $f(x) \neq 0$ . Then there exists an  $a$  such that\*

$$\lim_{x \rightarrow \infty}^* \frac{\ln |f(x)|}{\ln F(x - a)} = 0.$$

An analogous result can be formulated for the case when  $x \rightarrow -\infty$ . Theorem 3 should naturally be regarded as an analogue of the Ahlfors-Heins theorem for quasianalytic functions. It shows that the asymptotics of functions of the class  $C\{M_n\}$  is identical with the asymptotics along the real axis of functions of the class  $A$  in the strip  $D_a(F)$ .

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## CITED LITERATURE

<sup>1</sup> B. I. Korenblyum, *Studies on contemporary problems of the theory of functions of a complex variable*, Moscow, 1961.

<sup>2</sup> B. Ya. Levin, *Distribution of zeros of entire functions*, Moscow, 1956.

<sup>3</sup> I. I. Hirschman, *Am. J. Math.*, **72**, 1 (1950).

<sup>4</sup> H. Ahlfors, M. Heins, *Ann. Math.*, **50**, No. 2 (1949).

<sup>5</sup> C. Mandelbrojt, *Contiguous Series. Regularization of Sequences. Applications*, Moscow, 1955.

<sup>6</sup> C. E. Warschawski, *Mathematics*, Collected Translations, 2-4 (1958).

<sup>7</sup> I. I. Hirschman, *Am. J. Math.*, **72**, 2 (1950).

\* Theorem 3 sharpens a result obtained by Hirschman ((<sup>7</sup>), Theorem 3a).

*Note: Figure translations are in progress. See original paper for figures.*

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