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Abstract

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MATHEMATICS

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MULTIVALUED MAPPINGS AND SOME RELATED QUESTIONS

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Let X be a topological space. Denote by $A(X)$ the collection of all nonempty subsets of the space X . A single-valued mapping $\theta : Y \rightarrow A(X)$ is called a multivalued mapping of the space Y into X . A single-valued mapping $f : Y \rightarrow X$ is called a section for the multivalued mapping $\theta : Y \rightarrow A(X)$ if $fx \in \theta x$ for every point $x \in Y$ ⁽⁵⁻¹⁰⁾. We note that, unless a separation axiom is explicitly named, the space should be regarded as completely regular.

§ 1. **Continuous sections and σ -discrete sets.** Put

$$C(X) = \{L \in A(X) \mid L \text{ is bcompact}\}.$$

Theorem 1. Let $\theta : Y \rightarrow A(X)$ be a lower semicontinuous* mapping of a paracompact space Y into a complete metric space X . If the set

$$Y_1 = \{y \in Y \mid \theta y \text{ is not closed in } X\}$$

is σ -discrete in Y , then there exist multivalued mappings $\varphi : Y \rightarrow C(X)$ and $\psi : Y \rightarrow C(X)$ such that: a) $\varphi y \subseteq \psi y \subseteq \theta y$ for every point $y \in Y$; b) the mapping φ is lower semicontinuous; c) the mapping ψ is upper semicontinuous.

Theorem 1 allows us to strengthen a number of theorems from ^(5,6). In particular, we obtain:

Theorem 2. Let $\theta : Y \rightarrow A(X)$ be a lower semicontinuous mapping of a paracompact space Y into a Banach space X . If

$$Y_1 = \{y \in Y \mid \theta y \text{ is not closed in } X\}$$

is σ -discrete in Y , then there exists a continuous single-valued mapping $f : Y \rightarrow X$ such that: a) $fy \in \text{conv}(\theta y)$ for every point $y \in Y \setminus Y_1$; b) $fy \in \theta y$ for every point $y \in Y_1$.

Theorem 3. Let $\theta : Y \rightarrow A(X)$ be a lower semicontinuous mapping of a zero-dimensional paracompact space Y into a complete metric space X . If

$$Y_1 = \{y \in Y \mid \theta y \text{ is not closed in } X\}$$

is σ -discrete in Y , then there exists a continuous section $f : Y \rightarrow X$ for the mapping θ .

Corollary 1. Let $\theta : Y \rightarrow A(X)$ be a lower semicontinuous mapping of a σ -discrete paracompact space Y into a T_1 -space X with the first axiom of countability. Then there exists a continuous section $f : Y \rightarrow X$ for the mapping θ .

§ 2. **σ -continuous sections and dimension.** In ^(3,4) A. V. Arhangel'skii obtained a number of results on preservation of dimension under open and closed mappings. In doing so, A. V. Arhangel'skii used, in implicit form, the method of sections for single-valued mappings.

* A mapping $\theta : Y \rightarrow A(X)$ is lower (upper) semicontinuous if, for every open (closed) set A in X , the set

$$\theta^{-1}A = \{y \in Y \mid \theta y \cap A \neq \emptyset\}$$

is open (closed) in Y . Following V. Hurewicz (and in fact also Cauchy), V. I. Ponomarev in his fundamental works on multivalued mappings ⁽¹²⁻¹⁴⁾ calls mappings that are upper semicontinuous and, respectively, lower semicontinuous, continuous and, respectively, quasicontinuous.

A single-valued mapping $f : X \rightarrow Y$ is σ -continuous if $X = \bigcup_{i=1}^{\infty} X_i$ and $f|X_i$ is continuous for every natural number i . If, moreover, for every natural number i the set X_i is closed in X , then the mapping f is called F_σ -continuous.

Theorem 4. Let $\theta : Y \rightarrow A(X)$ be an upper semicontinuous isolated mapping* of the space Y into a regular space X with a countable net. Then there exists a σ -continuous selection $f : Y \rightarrow X$ for the mapping θ .

Theorem 5. Let $\theta : Y \rightarrow A(X)$ be a Y -closed mapping (see (7)) of a perfectly normal space Y into a metric space X . Suppose, further, that $\psi : Y \rightarrow A(X)$ is a lower semicontinuous isolated mapping such that $\psi y \subset \theta y$ for every point $y \in Y$. Then there exists an F_σ -continuous selection for the mapping θ .

Theorem 6. Let $\theta : Y \rightarrow A(X)$ be a lower semicontinuous mapping of a perfectly normal paracompact space Y into a complete metric space X . If

$$Y_1 = \{y \in Y \mid \theta y \text{ is not closed in } X \text{ and } \text{card } \theta y > \aleph_0\}$$

is σ -discrete in Y , then there exists an F_σ -continuous selection for the mapping θ .

A system $F_\pi(X)$ of subsets of a space X is called complete if there exists a space $\tilde{X} \supset X$, complete in the sense of Čech, such that $[L]_{\tilde{X}} = L$ for every $L \in F_\pi(X)$ (see (8,9)).

A topological space X is called a G_δ -space if $\{(x, x) \mid x \in X\}$ is a G_δ -set in $X \times X$.

Theorem 7. Let $\theta : Y \rightarrow F_\pi(X)$ be a Y -closed mapping of a perfectly normal space Y into a complete system $F_\pi(X)$ of a paracompact G_δ -space X . Suppose, further, that $\psi : Y \rightarrow A(X)$ is a lower semicontinuous isolated mapping such that $\psi y \subset \theta y$ for every point $y \in Y$. Then there exists an F_σ -continuous selection for the mapping ψ .

Theorem 8. Let $\theta : Y \rightarrow A(X)$ be a Y -closed lower semicontinuous isolated mapping of a perfectly normal space Y into a space X with a σ -discrete net. Then there exists an F_σ -continuous selection for the mapping θ .

We note that all symmetrizable spaces with the first axiom of countability and all spaces with a σ -discrete net are G_δ -spaces.

From Theorem 8 it follows that

Corollary 2. Let $f : X \rightarrow Y$ be an open-closed continuous and isolated mapping of a space X with a σ -discrete net onto Y . Then $\dim Y \leq \dim X$ and $\text{Ind } Y \leq \text{Ind } X$. Moreover, if X is weakly finite-dimensional or countably dimensional, then so is Y .

Theorem 7 allows us to obtain

Corollary 3. Let $f : X \rightarrow Y$ be a closed, inductively open and isolated mapping of a paracompact G_δ -space X onto a perfectly normal space Y . Then $\dim Y \leq \dim X$ and $\text{Ind } Y \leq \text{Ind } X$. Moreover, if X is weakly finite-dimensional or countably dimensional, then so is Y .

Theorem 9. Let the space X satisfy the weak first axiom of countability (see (1), p. 148). If X is the union of a countable number of closed symmetrizable subspaces, then X is symmetrizable.

Theorem 10. Let $\{U_\alpha \mid \alpha \in A\}$ be an open point-finite cover of a T_1 -space X . If each U_α is symmetrizable, then so is X .

Theorem 11. Let $\{F_\alpha \mid \alpha \in A\}$ be a closed locally finite cover of a space X by symmetrizable subspaces. Then X is symmetrizable.

Theorems 9 and 11 allow us to obtain

Corollary 4. Let $f : X \rightarrow Y$ be a closed, inductively open

* The mapping $\theta : Y \rightarrow A(X)$ is isolated if for every point $y \in Y$ there is an isolated point in θy (relative to θy).

and an isolated mapping of a paracompact symmetrizable space X with the first axiom of countability. Then Y is symmetrizable.

§ 3. Generalized sections. A k -metric space is a pair (X, d) , where X is a topological space and d is a pseudometric on X satisfying the following conditions: a) $H(x, d) = \{y \in X \mid d(y, x) = 0\}$ is a bicomactum; b) if $x \in [M]$,

then $d(x, M) = 0$; c) let Φ be a bicompactum and F a closed subset of X . If $d(x, y) > 0$ for any $x \in \Phi$ and $y \in F$, then also $d(\Phi, F) > 0$.

For any $L \subseteq X$ put $H(L, d) = \{x \in X \mid d(x, L) = 0\}$.

Theorem 12. Let $\theta : X \rightarrow A(Y)$ be a lower semicontinuous mapping of a paracompact space X into a k -metric space (Y, d) . Suppose, furthermore, that $F_\pi(Y) = \{\theta x \mid x \in X \setminus X_1\}$, where X_1 is a σ -discrete subset of X , is a complete system of the space Y . Then there exists an upper semicontinuous mapping $\psi : X \rightarrow C(Y)$ such that $\psi x \subseteq H(\psi x, d)$ and $\theta x \cap \psi x \neq \emptyset$ for every point $x \in X$.

Theorem 13. Let $f : X \rightarrow Y$ be an open continuous mapping of a p -space X onto a paracompact Y . If the system $\{f^{-1}y \mid y \in Y \setminus Y_1\}$, where Y_1 is a σ -discrete system in Y , is complete in X , then there exists such a closed subset $X_1 \subseteq X$ that $fX_1 = Y$ and $f|_{X_1}$ is a perfect mapping. In particular, Y is a p -space.

Corollary 5. Let $f : X \rightarrow Y$ be an open bicompact mapping of a p -space X onto Y . Then f is a k -covering mapping.

§ 4. The case of topological groups. Let γ be some uniform structure on X . Denote by $Z_\gamma(X)$ the collection of all γ -null-sets of the space X . Consider the system $A_\gamma(X)$ obtained as a result of applying the A -operation to the system $Z_\gamma(X)$. The sets of the system $A_\gamma(X)$ will be called γ - A -sets of the space X .

Let G be a topological group. In what follows, by γ we shall denote the left uniform structure of the group G .

Theorem 14. Let $f : G \rightarrow X$ be an open mapping of an almost metrizable* group G onto a separable Fréchet-Urysohn space X . If $\{f^{-1}x \mid x \in X\} \subseteq A_\gamma(G)$, then there are a bicompact subgroup H of countable character in G and an open mapping $\varphi : G/H \rightarrow X$ such that $f = \varphi \circ \pi$, where $\pi : G \rightarrow G/H$ is the natural mapping of G onto the quotient space G/H . If, moreover, $f^{-1}x$ is a finally compact subspace for every point $x \in X$, then the space X has a countable base and G is finally compact.

Corollary 6. Let $f : G \rightarrow X$ be an open bicompact mapping of an almost metrizable group G onto a regular separable space X with the first axiom of countability. Then X is metrizable.

Theorem 15. Let $f : G \rightarrow X$ be an open continuous mapping of an almost metrizable group G onto a locally separable metrizable space X . If $F_\pi(G) = \{f^{-1}x \mid x \in X\} \subseteq A_\gamma(G)$ and the system $F_\pi(G)$ is complete in G , then for every zero-dimensional subspace $X_1 \subseteq X$ there exists a subset $G_1 \subseteq G$ such that $fG_1 = X_1$ and $f|_{G_1}$ is a homeomorphism.

Theorem 16. Let $f : G \rightarrow X$ be an open continuous mapping of a locally bicompact group G onto a metrizable space X . Then for every zero-dimensional subspace $X_1 \subseteq X$ there exists a subset $G_1 \subseteq G$ such that $fG_1 = X_1$ and $f|_{G_1}$ is a homeomorphism.

§ 5. An approximation theorem for upper semicontinuous mappings.

Theorem 17. Let $\theta : X \rightarrow A(Y)$ be an upper semicontinuous mapping of a paracompact space X into a linearly topological space Y . Suppose, furthermore, that V_0 is some convex neighborhood of zero in Y . If for every point $x \in X$ there is a point $a(x) \in Y$ for which

* A group is almost metrizable if it contains a bicom pactum of countable character.

$a(x) + V_0 \ni \theta x$, then there exists such a continuous single-valued mapping $f : X \rightarrow Y$ that $fx + V_0 \ni \theta x$ for every point $x \in X$.

§ 6. Theorems on the nonexistence of sections. Let a continuous mapping $f : X \rightarrow Y$ be given. It is said that there exists a continuous section over a set $Y_1 \subseteq Y$ if there is a subset $X_1 \subseteq X$ such that $fX_1 = Y_1$ and $f|X_1$ is a homeomorphism. The nonexistence of continuous sections is connected either with the absence of zero-dimensionality, or with the absence of completeness of the inverse images of points.

Theorem 18. Let X be a compact space without isolated points. Then there exists a space Y with a countable base, which is the union of a countable number of compacta, and an open continuous finite-to-one mapping $f : Y \rightarrow X$ such that for every nonempty open subset $U \subseteq X$ there is no continuous section.

Theorem 19. Let X be a complete metrizable space without isolated points. Then there exist a metrizable space Y , which is the union of a countable number of closed and complete subspaces, and an open continuous finite-to-one mapping $f : Y \rightarrow X$ such that for every open nonempty subset $U \subseteq X$ there is no continuous section.

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