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1970

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Abstract

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UDC 517.946.9

MATHEMATICS

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BOUNDARY-VALUE PROBLEMS FOR SINGULAR ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

(Presented by Academician P. S. Novikov on 20 III 1970)

At present, boundary-value problems for elliptic equations of higher order that degenerate on the boundary of the domain are being intensively studied (³⁻⁵).

In the present note one more class of elliptic operators of higher order is studied, which degenerate on the corresponding hyperplane. The case when such degeneration occurs in a tangent plane was studied by the author earlier (²).

1. Let R_n^+ denote the half-space $x_n = y > 0$ of the Euclidean n -dimensional space R_n of points $x = (x', y)$, where $x' = (x_1, \dots, x_{n-1})$. We introduce for consideration a class of function spaces H_γ^l , a detailed description of which is given in (¹).

Let L be a linear differential operator of the form

$$L = \sum_{|l'|+2l_n \leq r} a_{l', l_n}(x) D_{x'}^{l'} B_y^{l_n}, \quad (1)$$

where the coefficients a_{l', l_n} are functions sufficiently smooth in $\overline{R_n^+}$, constant for sufficiently large values of x in modulus. Here and below B_y denotes the singular Bessel operator

$$\frac{\partial^2}{\partial y^2} + \frac{2\gamma}{y} \frac{\partial}{\partial y} \quad (\gamma > 0).$$

On the operator L at all points $x = x_0$ belonging to the domain $y \geq 0$ we impose the condition

$$\sum_{|l'|+2l_n=r} a_{l', l_n}(x) \xi^{l'} \xi_n^{2l_n} \neq 0 \quad \text{for } |\xi| \neq 0. \quad (2)$$

Such operators were called B -elliptic operators by the author ⁽²⁾. We note that a condition of type (2) was first formulated by the author in ⁽²⁾. An operator T is called a smoothing operator if it is a bounded operator from H_γ^s to H_γ^{s+1} . Let first the operator L have constant coefficients. Consider in all of R_n^+ the equation $Lu = L(D_{x'}, B_y)u(x) + f(x)$. Let L_0 be the principal part of the operator L .

Theorem 1. Let $u \in H_\gamma^s$, $s \geq 2m$, be a solution of the equation $L_0 u(x) = f(x)$. Then for $\gamma > 0$:

1) the a priori estimate is valid

$$\|u\|_{H_\gamma^s} \leq C \left(\|L_0 u\|_{H_\gamma^{s-2m}} + \|u\|_{H_\gamma^0} \right), \quad (3)$$

2) there exists a bounded operator R_0 , acting from H_γ^{s-2m} to H_γ^s , such that

$$L_0 R_0 = I + T. \quad (4)$$

With the aid of the preceding theorem the following is proved.

Theorem 2. Let $u \in H_\gamma^s$ and $s \geq 2m$. Let

$$L(x; D_{x'}, B_y) = \sum_{|l| \leq 2m} a_l(x) D_{x'}^{l'} B_y^{l_n}$$

be a B -elliptic operator with infinitely differentiable coefficients, bounded in all of R_n^+ , and with higher coefficients satisfying the condition: there exists an $\varepsilon > 0$ such that for all $x \in \overline{R}_n^+$,

$$|a_l(x) - a_l(0)| < \varepsilon.$$

Let, moreover, $a_l(x)$ satisfy the condition

$$|D_{x',y}^{l'+k} a_l(x)| \leq C y^{2r-i-k}, \quad (5)$$

where $k + i \leq 2r$, $1 \leq i \leq 2r - 1$, $l' + 2r \leq s - 2m$, $|l| \leq 2m - 1$. Then for $\gamma > 0$, for a solution of the equation $Lu = f$, the a priori estimate

$$\|u\|_{H_\gamma^s} \leq C \left(\|Lu\|_{H_\gamma^{s-2m}} + \|u\|_{H_\gamma^0} \right) \quad (6)$$

is valid, and there exists a bounded operator R_B , acting from H_γ^{s-2m} into H_γ^s , such that

$$LR_B = I + T.$$

We now consider a linear differential operator of the form

$$L(x; D_{x'}, D_y^2) = \sum_{|l| \leq 2m} a_l(x) D_{x'}^{l'} D_y^{2l_n}. \quad (7)$$

The conditions on the coefficients a_l of the operator L will be formulated below. We shall call the operator (7) B -elliptic if condition (2) is fulfilled for it in the domain $y \geq 0$. With the aid of Theorem 2 the following assertion is proved.

Theorem 3. Let $u \in H_\gamma^s$ and $s \geq 2m$. Let

$$L(x; D_{x'}, D_y^2) = \sum_{|l| \leq 2m} a_l(x) D_{x'}^{l'} D_y^{2l_n} \quad (8)$$

be a B -elliptic operator with infinitely differentiable coefficients, bounded in R_n^+ , and with higher coefficients satisfying the condition:

$$|a_l(x) - a_l(0)| < \varepsilon.$$

Let, moreover, $a_l(x)$ satisfy the conditions

$$\left| D_{x'}^{l'} \frac{\partial^k a_l(x)}{\partial y^k} \right| \leq C y^{2r-i-k}, \quad (9)$$

$$k \leq 2r - i, \quad 1 \leq i \leq 2r - 1, \quad l' + 2r \leq s - 2m, \quad |l| \leq 2m - 1;$$

$$|a_l(x)| \leq C_0 y^r, \quad 1 \leq r \leq 2m - 1, \quad |l| \leq 2m - 1; \quad (10)$$

$$\left| D_{x'}^{l'} \frac{\partial^{i-k-r} a_l(x)}{(y \partial y)^{i-k-r}} y^{i-k-r} \right| \leq C_1 y^{j+\tau}, \quad (11)$$

$$l' + i \leq s - 2m, \quad 1 \leq j \leq 2m - 1, \quad |l| \leq 2m, \quad l_n \neq 0.$$

Then for $\gamma < 0$, for a solution of the equation $Lu = f$ in R_n^+ , the estimate

$$\|u\|_{H_\gamma^s} \leq C \left(\|Lu\|_{H_\gamma^{s-2m}} + \|u\|_{H_\gamma^0} \right) \quad (12)$$

is valid.

and there exists a bounded operator R such that

$$LR = I + T, \quad (13)$$

where R acts from H_γ^{s-2m} into H_γ^s , and T is a smoothing operator.

The results obtained in this part allow us to pass to a bounded domain.

2. Consider a bounded domain Ω in the space R_n . Let the boundary $\partial\Omega$ of this domain be a smooth manifold. Denote by $\{V_{\nu_1}\}$ a covering of the boundary $\partial\Omega$ by a finite system of domains $V_1, \dots, V_r$ in R_n such that in each V_k there is defined an infinitely smooth change of coordinates $x \rightarrow y = y^{(k)}$ with positive Jacobian, under which the boundary $\partial\Omega$ is transformed locally into the hyperplane $y_n = 0$, while the part of the domain adjacent to $\partial\Omega$ is transformed into $y_n > 0$. In addition, it is assumed that under these changes of coordinates the normals to $\partial\Omega$ are transformed into normals to $y_n = 0$.

Let $\{V_{\nu_2}\}$ be such a finite system of open sets that

$$\Omega \setminus \bigcup_{\nu_1} V_{\nu_1} \subset \bigcup_{\nu_2} V_{\nu_2},$$

and, moreover,

$$\overline{\bigcup_{\nu_2} V_{\nu_2}} \cap \partial\Omega = \emptyset.$$

The space $H_\gamma^l(\Omega)$ is the set of functions defined in Ω for which the norm

$$\|u\|_{H_\gamma^l(\Omega)} = \sum_{\nu_1} \|\varphi_{\nu_1} u\|_{H_\gamma^l(R_n^+)} + \sum_{\nu_2} \|\varphi_{\nu_2} u\|_{H^l}, \quad (14)$$

is finite, where the norm $\|\varphi_{\nu_1} u\|_{H_\gamma^l}$ is computed in the local coordinates $y^{(\nu_1)}$ as the norm of a function defined in the corresponding half-space, and $\|\varphi_{\nu_2} u\|_{H^l}$ is the usual norm in the Sobolev space.

Let the operator L inside the domain Ω have the form

$$L(x, D_x) = \sum_{|\alpha| \leq r} a_\alpha(x) D_x^\alpha, \quad (15)$$

where the coefficients a_α are sufficiently smooth functions inside the domain Ω . We now define the operator L near the boundary $\partial\Omega$ of the domain Ω , i.e. at points

$$x \in \Omega \cap \left(\bigcup_{\nu_1} V_{\nu_1} \right).$$

We require of the operator L that, for any ν_1 , the operator $\tilde{L}_{\nu_1}$ be representable in the following form:

$$\tilde{L}_{\nu_1} = \sum_{|\alpha'|+2\alpha_n \leq r} a_{\alpha', \alpha_n}(y) D_y^{\alpha'} B_{y_n}^{\alpha_n}, \quad (16)$$

where a_{α', α_n} are sufficiently smooth functions defined in $\overline{R_n^+} \cap \tilde{V}_{\nu_1}$, and $\tilde{V}_{\nu_1}$ is the image of the set V_{ν_1} under the transformation $x \rightarrow y^{(\nu_1)}$. We impose the following restriction on the operator L . At every point $x \in \Omega$ the operator L is elliptic, i.e.

$$\sum_{|\alpha|=r} a_{\alpha}(x) \xi^{\alpha} \neq 0 \quad \text{for } |\xi| \neq 0, \quad \xi \in R_n, \quad x \in \Omega. \quad (17)$$

For any ν_1 , the operator $\tilde{L}_{\nu_1}$ satisfies, for $|\xi| \neq 0$, $\xi \in R_n$, $y \in \tilde{V}_{\nu_1} \cap \{y_n \geq 0\}$, the condition

$$\sum_{|\alpha'|+2\alpha_n=r} \tilde{a}_{\alpha', \alpha_n}(y) \xi^{\alpha'} \xi_n^{2\alpha_n} \neq 0. \quad (18)$$

An operator L satisfying these conditions will be called a B -elliptic operator in Ω .

Theorem 4. *Let the operator L be B -elliptic in the domain Ω . Suppose its coefficients satisfy the conditions of Theorem 2. Then, for $\gamma > 0$, the estimate*

$$\|u\|_{H_{\gamma}^s(\Omega)} \leq C \left(\|Lu\|_{H_{\gamma}^{s-2m}(\Omega)} + \|u\|_{H_{\gamma}^0(\Omega)} \right) \quad (19)$$

holds, and there exists a bounded operator R , acting from $H_{\gamma}^{s-2m}(\Omega)$ into $H_{\gamma}^s(\Omega)$, such that

$$LR = I + T, \quad (20)$$

where I is the identity operator and T is a smoothing operator.

An analogous assertion holds for the operator $\bar{L}$, defined inside the domain Ω by means of the operator (15), and near the boundary by means of an operator of the form

$$\tilde{L}_{\nu_1} = \sum_{|\alpha'|+2\alpha_n=r} \tilde{a}_{\alpha',\alpha_n}(y) D_y^{\alpha'} D_y^{2\alpha_n}, \quad (21)$$

whose coefficients satisfy, near the boundary, the conditions of Theorem 3. The operators $\bar{L}$ and $\tilde{L}_{\nu_1}$ themselves satisfy conditions (17) and (18), respectively.

Using the Fourier-Bessel transform, one can show that the smoothing operators T thus obtained are completely continuous for a bounded domain Ω . Hence the operators L and $\bar{L}$, acting respectively from H_γ^{2m+k} into H_γ^k , are Noetherian.

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Received
19 III 1970

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