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Abstract

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MATHEMATICS

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ON SUFFICIENT CONDITIONS FOR THE UNIVALENCE OF SOLUTIONS OF INVERSE BOUNDARY-VALUE PROBLEMS

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The present note continues the investigations of V. S. Rogozhin, L. A. Aksent'ev, and S. N. Kudryashov (see the bibliography in ⁽¹⁾). It gives sufficient conditions for the univalence of solutions of two inverse boundary-value problems ([¹], Ch. 1; [²], §33) on the basis of the solution of certain problems in the theory of univalent functions.

§ 1. In Theorems 1-5 it is assumed that the function $f(z)$ is regular in the disk $|z| < 1$, while the function $F(\zeta)$ is regular in the domain $|\zeta| > 1$, except for the point $\zeta = \infty$, where it has a simple pole.

Theorem 1. *In order that the function $F(\zeta)$ be univalent in $|\zeta| > 1$, it is necessary that*

$$\left| \frac{F''(\zeta)}{F'(\zeta)} \right| \leq \frac{6}{|\zeta|^3 - |\zeta|}, \quad |\zeta| > 1; \quad (1)$$

sufficient that

$$\left| \frac{F''(\zeta)}{F'(\zeta)} \right| \leq \frac{2(\sqrt{5} - 2)}{|\zeta|^3 - |\zeta|}, \quad |\zeta| > 1. \quad (2)$$

Inequality (1) follows from the theorem of G. M. Goluzin (³, p. 139). The proof of the sufficiency of condition (2) is carried out analogously to the proof of the corresponding result in (⁴) for $f(z)$.

Theorem 2. *The function $F(\zeta)$ is univalent in the domain $|\zeta| > 1$ if one of the following conditions is satisfied:*

$$1^\circ. \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{F''(\zeta)}{F'(\zeta)} \right|^2 d\theta \leq \frac{8}{9}, \quad \zeta = \rho e^{i\theta}, \quad \rho > 1. \quad (3)$$

$$2^\circ. \quad \left| \frac{F''(\zeta)}{F'(\zeta)} \right| \leq \frac{4}{3}, \quad |\zeta| > 1. \quad (4)$$

The proof is based on the following extension of V. V. Pokornyi's sufficient condition for univalence ⁽⁵⁾ to the domain $|\zeta| > 1$, by means of the change of variable $\zeta = 1/z$.

Lemma. *In order that a meromorphic function $F(\zeta)$ in $|\zeta| > 1$ be univalent in $|\zeta| > 1$, it is sufficient that, for some λ ($0 \leq \lambda \leq 2$), the inequality*

$$|\{F, \zeta\}| \equiv \left| \left(\frac{F''(\zeta)}{F'(\zeta)} \right)' - \frac{1}{2} \left(\frac{F''(\zeta)}{F'(\zeta)} \right)^2 \right| \leq \frac{C(\lambda)}{|\zeta|^4 (1 - |\zeta|^{-2})^\lambda}, \quad |\zeta| > 1, \quad (5)$$

holds, where

$$C(\lambda) = \begin{cases} 2^{3\lambda-1} \pi^{2(1-\lambda)}, & 0 \leq \lambda \leq 1, \\ 2^{3-\lambda}, & 1 \leq \lambda \leq 2. \end{cases}$$

By assumption, the function $F(\zeta)$, satisfying the conditions of the theorem, is representable in $|\zeta| > 1$ by the series

$$F(\zeta) = a\zeta + a_0 + a_1/\zeta + a_2/\zeta^2 + \dots, \quad a \neq 0.$$

Therefore, when (3) or (4) is fulfilled, the function regular in $|\zeta| > 1$,

$\varphi(\zeta) \equiv F''(\zeta)/F'(\zeta)$, is representable by the series

$$\varphi(\zeta) = \sum_{k=3}^{\infty} c_k \zeta^{-k}, \quad |\zeta| > 1.$$

1°. Using the Cauchy-Bunyakovsky inequality, we have

$$|\varphi(\zeta)|^2 = \left| \sum_{k=3}^{\infty} c_k \zeta^{-k} \right|^2 \leq \sum_{k=3}^{\infty} |c_k|^2 \cdot \sum_{k=3}^{\infty} |\zeta|^{-2k}, \quad |\zeta| > 1,$$

$$|\varphi'(\zeta)|^2 = \left| \sum_{k=3}^{\infty} k c_k \zeta^{-k-1} \right|^2 \leq \sum_{k=3}^{\infty} |c_k|^2 \cdot \sum_{k=3}^{\infty} k^2 |\zeta|^{-2k-2}, \quad |\zeta| > 1.$$

Condition (3), after passage to the limit as $|\zeta| \rightarrow 1$, gives

$\sum_{k=3}^{\infty} |c_k|^2 \leq \frac{8}{9}$, and therefore

$$|\varphi(\zeta)|^2 \leq \frac{8}{9} \frac{1}{|\zeta|^6(1-|\zeta|^{-2})}, \quad |\varphi'(\zeta)|^2 \leq \frac{8}{9} \frac{9-11|\zeta|^{-2}+4|\zeta|^{-4}}{|\zeta|^8(1-|\zeta|^{-2})^3}, \quad |\zeta| > 1.$$

Noting that the Schwarzian

$\{F, \zeta\} \equiv \varphi'(\zeta) - \frac{1}{2}\varphi^2(\zeta)$, we obtain the estimate

$$|\{F, \zeta\}| \leq |\varphi'(\zeta)| + \frac{1}{2}|\varphi(\zeta)|^2 \leq \frac{2\sqrt{2}}{3} S(t) \frac{1}{|\zeta|^4(1-|\zeta|^{-2})^{3/2}},$$

where

$$S(t) = \sqrt{9-11t+4t^2} + \frac{\sqrt{2}}{3} t\sqrt{1-t}, \quad t = |\zeta|^{-2}, \quad |\zeta| > 1.$$

But $S'(t) < 0$, $0 \leq t \leq 1$; hence $S(t) \leq S(0) = 3$. Then the Schwarzian $\{F, \zeta\}$ satisfies inequality (5) of the lemma for $\lambda = 3/2$, which proves the univalence of the function $F(\zeta)$.

2°. Condition (4) is equivalent to the inequality $|\varphi(\zeta)| \leq 4/3$. By the theorem of G. M. Goluzin ([3], p. 325), applied to the function $\varphi(z^{-1})$ in the disk $|z| < 1$, taking into account the order of the zero at the point $z = 0$, we shall have

$$\left| \varphi\left(\frac{1}{z}\right) \right| \leq \frac{4}{3}|z|^3, \quad \left| -\frac{1}{z^2}\varphi'\left(\frac{1}{z}\right) \right| \leq \frac{4}{3} \frac{3|z|^2}{1-|z|^6}, \quad |z| < 1,$$

or

$$|\varphi(\zeta)| \leq \frac{4}{3|\zeta|^3}, \quad |\varphi'(\zeta)| \leq \frac{4}{|\zeta|^4(1-|\zeta|^{-6})}, \quad |\zeta| > 1.$$

The obtained inequalities allow us to estimate the Schwarzian

$$|\{F, \zeta\}| \leq |\varphi'(\zeta)| + \frac{1}{2}|\varphi(\zeta)|^2 \leq \frac{4}{|\zeta|^4(1-|\zeta|^{-2})}, \quad |\zeta| > 1,$$

i.e., the function $F(\zeta)$ satisfies condition (5) of the lemma for $\lambda = 1$, and consequently is univalent.

§ 2. Let $f'(z) \neq 0$ in $|z| < 1$ and $F'(\zeta) \neq 0$ in $|\zeta| > 1$. Then the integrals

$$\int [f'(z)]^\alpha dz, \quad |z| < 1; \quad \int [F'(\zeta)]^\alpha d\zeta, \quad |\zeta| > 1,$$

where α is a complex constant, define certain analytic functions $f_\alpha(z)$ and $F_\alpha(\zeta)$ in the domains $|z| < 1$ and $|\zeta| > 1$, respectively. For the functions $f_\alpha(z)$ and $F_\alpha(\zeta)$ the following assertions are valid.

Theorem 3. *If $f(z)$ and $F(\zeta)$ are univalent, then the functions $f_\alpha(z)$ and $F_\alpha(\zeta)$ are also univalent for every complex α satisfying the inequality*

$$3|\alpha| + 9|\alpha - \alpha^2| \leq 1.$$

We note that, for the function $f(z)$, Theorem 3 includes the result from paper ⁽⁴⁾, coinciding with it for real and negative α . The following two theorems develop the ideas of the indicated paper ⁽⁴⁾ for certain subclasses of univalent functions.

Theorem 4. *If the function $f(z)$ is univalent and maps the disk $|z| < 1$ onto a convex domain, then the function $f_\alpha(z)$ is univalent for any complex α satisfying the inequality*

$$|\alpha| + 4|\alpha - \alpha^2| \leq 1;$$

for real α it is sufficient to require only that the inequality

$$-\frac{1}{2} \leq \alpha \leq 1$$

hold.

Theorem 5. *If the function $F(\zeta)$ is univalent and maps $|\zeta| > 1$ onto a domain with convex complement, then the function $F_\alpha(\zeta)$ is univalent for any complex α satisfying the inequality*

$$3|\alpha| + 4|\alpha - \alpha^2| \leq 1.$$

The theorems formulated above are obtained using the known necessary conditions for univalence (⁽³⁾, pp. 52, 139, 510, 582)) and Nehari's results ⁽⁶⁾.

§ 3. Let

$$f(z) = \int \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right\} dz, \quad |z| < 1, \quad (6)$$

where $P(\theta)$ is a real 2π -periodic function.

Theorem 6. The function (6) is univalent if one of the following conditions is fulfilled:

1°. The function $P(\theta)$ is continuous and, for any θ_1 and θ_2 , $-\pi < \theta_1 < \theta_2 \leq \pi$, the inequality

$$-\pi < (\theta_2 - \theta_1) + \frac{P(\theta_1) - P(\theta_2)}{a} < 3\pi$$

holds, or the stronger condition

$$|P(\theta_1) - P(\theta_2)| < \pi|a|,$$

where a is one of the real roots of the equation

$$3|a| + 9|ai + a^2| = 1.$$

2°. There exists a continuous derivative $P'(\theta)$ of the function $P(\theta)$, essentially bounded on one side,

$$P'(\theta) > -|b|$$

or

$$P'(\theta) < |b|,$$

where b is a real root of the equation

$$|b| + 4|bi + b^2| = 1.$$

The proof is carried out by applying Theorems 3 and 4 to the auxiliary function $g(z)$, defined by the equality $[g'(z)]^\alpha = f'(z)$. For $\alpha = ai$ ($\alpha = \pm bi$), it follows from condition 1° (2°) that the function $g(z)$ is almost convex (convex) in the disk $|z| < 1$.

Let

$$F(\zeta) = \int \exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta) \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\theta \right\} d\zeta, \quad |\zeta| > 1, \quad (7)$$

where the real 2π -periodic function $P(\theta)$ satisfies the condition

$$\int_{-\pi}^{\pi} P(\theta) e^{i\theta} d\theta = 0. \quad (8)$$

Theorem 7. The function $F(\zeta)$, defined by the integral representation (7) with the additional condition (8), is univalent if one of the following conditions is fulfilled:

1°. The function $P(\theta)$ is continuous and has bounded oscillation

$$|P(\theta_1) - P(\theta_2)| < A,$$

where A is the real root of the equation $\operatorname{sh} A = (\sqrt{5} - 2)/2$.

2°. There exists a continuous derivative $P'(\theta)$ of the function $P(\theta)$, essentially bounded on one side, $P'(\theta) > -|c|$ or $P'(\theta) < |c|$, where c is the real root of the equation $3|c| + 4|ci + c^2| = 1$.

3°. The function $P(\theta)$ satisfies the Lipschitz condition

$$|P(\theta_1) - P(\theta_2)| \leq K|\theta_1 - \theta_2|, \quad \text{where } K \leq 2/3.$$

From conditions 1° and 3° it follows that $F(\zeta)$ satisfies inequalities (2) or (3), respectively, and therefore is univalent. If 2° is fulfilled, the function $\Phi(\zeta)$ ($[\Phi'(\zeta)]^{\pm ci} = F'(\zeta)$) maps $|\zeta| > 1$ onto a domain with convex complement. The univalence of $F(\zeta)$ in this case follows from Theorem 5 applied to $\Phi(\zeta)$ with $a = \pm ci$.

Let us draw some conclusions. Conditions 1° of Theorems 6 and 7 show that $P(\theta)$ need not possess differentiability properties. Conditions 2° constitute a first attempt to restrict the interval of variation of $P'(\theta)$ by half-infinite intervals, whereas in the known univalence conditions expressed in terms of $P'(\theta)$ it was essentially assumed that $|P'(\theta)| \leq M < \infty$ (see, for example, (1), p. 56). Condition 3° of the last theorem was previously proved ((7), p. 16) only for $K \leq 0, 2, \dots$

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