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EXISTENCE OF LACUNARY SYSTEMS IN BANACH SPACES

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1970

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Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

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EXISTENCE OF LACUNARY SYSTEMS IN BANACH SPACES

(Presented by Academician V. I. Smirnov on 27 V 1969)

1. Let a separable Banach space E be densely embedded in a Hilbert space H , $E \neq H$, and $\|x\|_H \leq \|x\|_E$ for all $x \in E$. After the natural identifications one may assume that $E \subset H \subset E^*$.

Let $(x_k) \subset E$ be a Riesz basic sequence in H , i.e., a Riesz basis in the closure $[x_k]_H$ (with respect to the norm in H) of its linear span. This system is called E -lacunary ⁽¹⁾ if, for every sequence $(a_k) \in l^2$, the series

$$\sum_{k=1}^{\infty} a_k x_k$$

converges in the norm of the space H to some element of the space E .

In a paper of S. Banach ⁽²⁾ (see also ⁽³⁻⁵⁾) it was proved that if $(e_k) \subset L^p$ ($2 < p < +\infty$) is an orthonormal system with a bounded sequence of norms, then one can extract from it an L^p -lacunary subsystem.

The purpose of the present paper is to prove the existence of lacunary systems in Banach spaces whose norm is twice continuously Fréchet differentiable. We note that in any L^p , for $p > 2$, as follows from results of Sundaresan ⁽⁸⁾, the norm is twice continuously differentiable.

2. For simplicity we shall regard E as a Banach space over the field R of real numbers. As is known ⁽⁶⁾, p. 144; see also ^(7,8), in order that the function $\varphi(x) = \|x\|$ be differentiable in the sense of Gâteaux at the point $x \in E$ in any direction $h \in E$, it is necessary and sufficient that there exist a unique functional $f_x \in S^* = \{f \in E^* \mid \|f\| = 1\}$ such that $f_x(x) = \|x\|$. In this case the Gâteaux derivative $\varphi'(x)$ of this function is equal to f_x . If the norm $\|x\|$ is Fréchet differentiable at the point x , then it is also Gâteaux differentiable and the two derivatives are equal ⁽⁹⁾, p. 61). If the norm $\|x\|$ is Fréchet differentiable in $E \setminus \{0\}$, then the function $\psi : x \mapsto \|x\|^2$ is also differentiable ⁽¹⁰⁾, p. 174), and

$$(\|x\|^2)' = 2\|x\|f_x.$$

In what follows we shall assume that the norm is twice continuously Fréchet differentiable for all $x \neq 0$. Then

$$\|x + h\| = \|x\| + f_x(h) + T_x(h, h) + r_2(x; h), \quad \lim_{\|h\| \rightarrow 0} \frac{r_2(x; h)}{\|h\|^2} = 0.$$

Obviously, the second derivative of the norm is the second successive derivative

$$(\|x\|)'' = T_x = [(\|x\|)']' = f'_x.$$

Since

$$(\|x\|^2)'' = (2\|x\|f'_x)' = 2f'_x(\cdot)f'_x(\cdot) + 2\|x\|f''_x,$$

then by Taylor's formula ((10), p. 218)

$$\|x + h\|^2 = \|x\|^2 + 2\|x\|f'_x(h) + [f'_x(h)]^2 + \|x\|f''_x(h, h) + \omega_2(x; h),$$

$$\lim_{\|h\| \rightarrow 0} \frac{\omega_2(x; h)}{\|h\|^2} = 0.$$

We shall agree to call the second derivative bounded on a set A if, for every $h \in E$,

$$\sup_{x \in A} |f'_x(h, h)| < +\infty.$$

Lemma 1. *If in a Banach space E the norm is twice continuously Fréchet differentiable and its second derivative is bounded on the unit sphere, then for any $x \in E$ and $h \in E$ the inequality*

$$\|x + h\|^2 \leq \|x\|^2 + 2\|x\|f'_x(h) + M\|h\|^2,$$

holds, where the number M does not depend on the choice of the elements x and h .

Proof. Since (8) $f'_{\lambda x} = \frac{1}{|\lambda|} f'_x$, it follows that $\|x\|f'_x = f'_{x/\|x\|}$. Therefore, for each $x \in E$, taking into account the continuity of f'_x , we have

$$\sup_{x \in E} \|\|x\|f'_x(h, h)\| < +\infty, \quad x \neq 0.$$

But then, by (11), there exists a number $M_1 > 0$ such that

$$\|\|x\|f'_x(h, h)\| \leq M_1\|h\|^2.$$

Let us estimate the remainder term $\omega_2(x; h)$ in Taylor's formula. We have (10)

$$\begin{aligned} |\omega_2(x; h)| &= \left| \int_0^1 (1-t) [f_{x+th}^2(h) + f'_{x+th}(h, h)] dt \right| \leq \\ &\leq \frac{1}{4}(\|h\|^2 + M_1\|h\|^2) = M_2\|h\|^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \|x + h\|^2 &\leq \|x\|^2 + 2\|x\|f_x(h) + \|h\|^2 + M_1\|h\|^2 + M_2\|h\|^2 = \\ &= \|x\|^2 + g_x(h) + M\|h\|^2, \end{aligned}$$

where $M = 1 + M_1 + M_2$ and $g_x(h) = \|x\|f_x(h)$.

Theorem 1. *Let a reflexive space E , with a twice continuously Fréchet differentiable norm whose second derivative is bounded on the unit sphere, be densely embedded in a Hilbert space H . If the system $(e_k) \subset E$ is orthonormal in H and the norms of the elements of the system are uniformly bounded in E , then from this system one can extract an E -lacunary subsystem.*

The proof given below is carried out by a method analogous to the method of proof in the paper (2) (see also (4), p. 287).

Proof. As we have already noted, we may assume that $E \subset H \subset E^*$. Obviously, H is densely embedded in E^* . Otherwise, by the reflexivity of the space E , there would be an element $x_0 \neq 0$ such that, for all $y \in H$, $(x_0, y) = 0$, which is impossible. From the conditions of the theorem it follows that there is a number $\alpha > 0$ such that

$$\|e_n\|_E \leq \alpha, \quad \lim_{n \rightarrow \infty} (e_n, y) = 0 \quad \text{for } y \in H.$$

Consequently, the sequence (e_n) converges weakly to zero in the space E . Therefore, for every $x \in E$,

$$\lim_{n \rightarrow \infty} g_x(e_n) = 2 \lim_{n \rightarrow \infty} \|x\|f_x(e_n) = 0.$$

Let

$$\beta \in S_r^2 = \left\{ (\beta_i)_{i=1}^r \left| \sum_{i=1}^r \beta_i^2 \leq 1 \right. \right\}, \quad x_\beta = \sum_{i=1}^r \beta_i e_i.$$

We shall prove that

$$\lim_{n \rightarrow \infty} g_{x_\beta}(e_n) = 0$$

uniformly with respect to $\beta \in S_r^2$. Indeed, if for some $\varepsilon > 0$ there were a sequence $(\beta^{(k)}) \subset S_r^2$ such that $g_{x_{\beta^{(k)}}}(e_k) \geq \varepsilon$, then, choosing a convergent subsequence, which we again denote by $(\beta^{(k)})$, with limit point $\beta^{(0)}$, we would obtain for this point

$$\lim_{n \rightarrow \infty} g_{x_{\beta^{(0)}}}(e_n) = 0.$$

Taking into account that, for fixed r ,

$$\lim_{k \rightarrow \infty} x_{\beta^{(k)}} = \lim_{k \rightarrow \infty} \sum_{i=1}^r \beta_i^{(k)} e_i = \sum_{i=1}^r \beta_i^{(0)} e_i = x_{\beta^{(0)}},$$

we have, by virtue of the assumed continuity of the derivative f'_x ,

$$\lim_{k \rightarrow \infty} g_{x_{\beta^{(k)}}} = g_{x_{\beta^{(0)}}}.$$

Consequently,

$$\lim_{k \rightarrow \infty} g_{x_{\beta^{(k)}}}(e_k) = 0,$$

which contradicts the assumption. Thus,

$$g_{x_\beta}(e_n) < \frac{1}{r} \quad \text{for } n \geq N(r) \text{ for every } \beta \in S_r^2.$$

We now choose a subsequence $(e_{r_n})_{n=1}^\infty$ in the following way. Put $r_1 = 1$, and if r_n has already been chosen, set $r_{n+1} = \max(1 + r_n; N(r_n))$. With this choice $r_n \geq n$. We shall show that the chosen subsystem is E -lacunary. Put

$$S_n = \sum_{i=1}^n a_i e_{r_i}; \quad n = 1, 2, \dots; \quad \sum_{i=1}^\infty |a_i|^2 \leq 1.$$

Applying Lemma 1, we obtain

$$\|S_{n+1}\|_E^2 = \|S_n + a_{n+1} e_{r_{n+1}}\|_E^2 \leq \|S_n\|_E^2 + g_{S_n}(a_{n+1} e_{r_{n+1}}) +$$

$$+M\|a_{n+1}e_{r_{n+1}}\|_E^2 \leq \|S_n\|_E^2 + \frac{1}{n}|a_{n+1}| + a^2M|a_{n+1}|^2.$$

Adding these inequalities for $n = 1, 2, \dots, m-1$, we obtain

$$\|S_m\|_E^2 \leq \|S_1\|_E^2 + \sum_{n=1}^{m-1} \frac{1}{n}|a_{n+1}| + a^2M \sum_{n=1}^{m-1} |a_{n+1}|^2 \leq l.$$

If, however, the sequence $(a_i) \in l^2$ is arbitrary, then

$$\left\| \sum_{k=1}^m a_k e_{r_k} \right\|_E^2 \leq l \sum_{k=1}^m |a_k|^2; \quad m = 1, 2, \dots$$

The resulting inequality proves the E -lacunarity (1) of the system (e_{r_n}) .

Lemma 2. If in the space E the norm is twice continuously Fréchet differentiable and its second derivative is bounded on the unit sphere, then for any $x \in E$ and $y \in E$ there exists a number $\eta = \pm 1$ such that

$$\|x + \eta y\|^2 \leq \|x\|^2 + M\|y\|^2,$$

where the number M depends neither on x nor on y .

The proof of Lemma 2 differs from the proof of Lemma 1 only in that, choosing the number $\eta = \pm 1$ so that $f'_x(\eta y) < 0$, we can remove this term of Taylor's formula without violating the inequality of Lemma 1.

Lemma 3. For any elements $y_1, y_2, \dots, y_n$, one can choose numbers

$\eta_k = \pm 1$, $k = 1, 2, \dots$, such that

$$\left\| \sum_{i=1}^n \eta_i y_i \right\|_E^2 \leq M \sum_{i=1}^n \|y_i\|_E^2,$$

where M is a number independent of either the choice of the elements or of their number n .

The proof of the lemma is easily obtained by the method of mathematical induction, using Lemma 2.

Theorem 2. Let E be a reflexive space in which the norm is twice continuously Fréchet differentiable and its second derivative is bounded on the unit sphere. If $(x_k) \subset E$ is an unconditional basic sequence for which

$$0 < \inf \|x_n\|_E \leq \sup \|x_n\|_E < +\infty$$

and $\|x_n\|_E \leq a\|x_n\|_H$, then the system (x_n) is equivalent to the unit basis in l^2 .

Proof. Assume for simplicity that $\|x_n\|_E = 1$.

If the series $\sum_{k=1}^{\infty} \alpha_k x_k$ converges in E , then it converges, moreover unconditionally, in the space H . Therefore

$$\sum_{k=1}^{\infty} \|\alpha_k x_k\|_H^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 \|x_k\|_H^2 < +\infty,$$

and, consequently, according to the conditions of the theorem, the series converges

$$\sum_{k=1}^{\infty} |\alpha_k|^2. \quad (1)$$

Conversely, if the series (1) converges, then by Lemma 3 there exist numbers $\eta_k = \pm 1$ such that, for any $m = 1, 2, \dots$,

$$\left\| \sum_{k=1}^m \eta_k \alpha_k x_k \right\|_E \leq \sqrt{M} \left(\sum_{k=1}^m \|\alpha_k x_k\|_E^2 \right)^{1/2} = \sqrt{M} \left(\sum_{k=1}^m |\alpha_k|^2 \right)^{1/2}.$$

Taking into account the reflexivity of the space E , one may assert¹² that the basis (x_k) is boundedly complete in the space $[x_k]E$, and therefore

$$\sum_{k=1}^{\infty} \eta_k \alpha_k x_k$$

converges, and since the basis (x_k) is unconditional, the series

$$\sum_{k=1}^{\infty} \alpha_k x_k$$

also converges.

Corollary 1. If, under the conditions of Theorem 2, there exists in E an unconditional basis (x_n) such that

$$0 < \inf \|x_n\|_E \leq \sup \|x_n\|_E < +\infty,$$

then

$$\liminf_{n \rightarrow \infty} \|x_n\|_H = 0.$$

Corollary 2. If, under the conditions of Theorem 2, $(e_n) \subset E$ is an orthonormal system in H and an unconditional basis in E , then

$$\limsup_{n \rightarrow \infty} \|e_n\|_E = +\infty,$$

$$\liminf_{n \rightarrow \infty} \|e_n\|_{E^*} = 0.$$

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Received

13 V 1969

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Note: Figure translations are in progress. See original paper for figures.

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