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T. M. Fedulova

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Abstract

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MATHEMATICS

T. M. FEDULOVA

APPLICATION OF THE DISPERSION METHOD IN ADDITIVE PROBLEMS WITH A RE- STRICTED SET OF PRIME NUMBERS

(Presented by Academician Yu. V. Linnik on 23 VII 1969)

1. The dispersion method, developed by Yu. V. Linnik ⁽¹⁾, is applicable to the solution of various binary additive problems of the form

$$\alpha + \beta = n, \quad (1)$$

$$\alpha - \beta = a, \quad (2)$$

where α and β run through prescribed sequences of natural numbers, sufficiently well distributed in arithmetic progressions; $a \neq 0$ is a fixed integer; n is a sufficiently large natural number ($n \rightarrow \infty$), $\beta < n$ (see ⁽³⁾).

In particular, the dispersion method makes it possible to find an asymptotic formula for the number of solutions of equations (1) and (2) in the case when α runs through the sequence of natural numbers with a restricted number of prime divisors, while β takes the values of a given quadratic form.

Equations (1) and (2) can be written in a uniform way in the form

$$\alpha + (-1)^{j-1}\beta = n_j \quad (j = 1, 2), \quad (3)$$

where

$$n_j = \begin{cases} n, & \text{if } j = 1, \\ a, & \text{if } j = 2. \end{cases}$$

The aim of this note is to investigate equation (3) for $\alpha = p_1 p_2 \dots p_k$, where p_i are prime numbers and k is a fixed natural number. In the case when $\beta = xy$ and $\beta = x^2 + y^2$, equation (3) is solved by the dispersion method. If $\beta = \varphi(x, y)$, where $\varphi(x, y)$ is a positive definite quadratic form with discriminant different

from a perfect square, then the asymptotic formula for the number of solutions of equation (3) is found by a combined application of the dispersion method and elementary ergodic considerations (see ^(2, 4)).

The results of the note are a generalization of investigations by Yu. V. Linnik ⁽¹⁾, B. M. Bredikhin ^(3, 5, 6), and A. A. Polyanskii ⁽⁷⁾ on the problems of Titchmarsh and Hardy-Littlewood.

In what follows we shall assume that $\alpha = p_1 p_2 \dots p_k$ consists of distinct prime numbers, and moreover $(\alpha, n_j) = 1$. The general case reduces to this one.

2. Let us formulate the main results. Let $Q_j(n)$ be the number of solutions of the equation

$$p_1 p_2 \dots p_k + (-1)^{j-1} xy = n_j \quad (j = 1, 2), \quad (4)$$

where $p_1, p_2, \dots, p_k$ independently run through distinct prime numbers subject to the condition $(p_1 p_2 \dots p_k, n_j) = 1$, k is a fixed number, and x and y independently run through natural numbers, $xy < n$.

Theorem 1. As $n \rightarrow \infty$,

$$Q_j(n) = \frac{315\zeta(3)}{2\pi^4} \prod_{p|n_j} \frac{(p-1)^2}{p^2 - p + 1} A_k(n, n_j) n + O\left(\frac{n}{(\ln n)^{1-\varepsilon_1}}\right), \quad (5)$$

where

$$A_k(n, n_j) = \sum_{l=0}^{k-1} c_k^l \sum_{\substack{\delta_1/n_j \\ 0 \leq \omega(\delta_1) \leq l}} \frac{\mu(\delta_1)}{\delta_1} \sum_{\substack{(\delta_2, n_j)=1 \\ \omega(\delta_2)=l-\omega(\delta_1)}} \frac{\mu(\delta_2)}{\varphi^2(\delta_2)} \sum_{r=0}^{k-l-1} \frac{B_r}{(k-l-r-1)!} \times (\ln \ln n)^{k-l-r-1}, \quad (6)$$

B_r are certain constants. The numbers B_r appear in the asymptotic evaluation of sums of the form:

$$\sum_{p_1 p_2 \dots p_s < n} 1 \quad (\text{see (8)});$$

$\omega(\delta)$ is the number of distinct prime divisors of δ .

Let $S_j(n)$ be the number of solutions of the equation

$$p_1 p_2 \dots p_k + (-1)^{j-1} (x^2 + y^2) = n_j, \quad (j = 1, 2), \quad (7)$$

where all the conditions from (4) are retained and $x^2 + y^2 < n$.

Theorem 2. As $n \rightarrow \infty$,

$$S_j(n) = \pi \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)} \right) \prod_{p/n_j} \frac{(p-1)(p-\chi_4(p))}{p^2-p+\chi_4(p)} B_k(n, n_j) \frac{n}{\ln n} + O\left(\frac{n}{(\ln n)^{1+\gamma}}\right), \quad (8)$$

where

$$B_k(n, n_j) = \sum_{l=0}^{k-1} c_k^l \sum_{\substack{\delta_1/2n_j \\ 0 \leq \omega(\delta_1) \leq l}} \frac{\mu(\delta_1)}{\delta_1} \sum_{\substack{(\delta_2, 2n_j)=1 \\ \omega(\delta_2)=l-\omega(\delta_1)}} \frac{\mu(\delta_2)}{\delta_2} \sum_{r=0}^{k-l-1} \frac{B_r}{(k-l-r-1)!} \times (\ln \ln n)^{k-l-r-1}, \quad \gamma = 0.042. \quad (9)$$

Let us note that the numbers $A_k(n, n_j)$ and $B_k(n, n_j)$ are equal to 1 for $k = 1$. Thus, the theorems of Yu. V. Linnik (1), which provide a solution of the Titchmarsh and Hardy-Littlewood problems, and the analogue of the Titchmarsh problem considered in the work of B. M. Bredikhin (3), are obtained as special cases of Theorems 1 and 2 for $k = 1$.

Let $Q(n)$ be the number of solutions of the equation

$$p_1 p_2 \dots p_k + \varphi(x, y) = n, \quad (10)$$

where $\varphi(x, y)$ is a positive definite quadratic form with discriminant different from a perfect square, and the conditions imposed on the numbers p_i in (4) are retained.

Theorem 3. As $n \rightarrow \infty$,

$$Q(n) = \sigma_k(n, \varphi) \frac{n}{\ln n} + O\left(\frac{n}{(\ln n)^{1+\gamma_1}}\right), \quad (11)$$

where $\sigma_k(n, \varphi) \gg 1/(\ln \ln n)^C$ is a complicated arithmetic factor, $0 < \gamma_1 < \gamma$.

3. Let us consider brief outlines of the proofs of Theorems 1-3.

- 1) We transform $Q_j(n)$, using the symmetry of equation (4) with respect to x and y , as well as upper estimates for $\pi_k(n)$ (8). It is easy to see that

$$Q_j(n) = 2\tilde{Q}_j(n) + O(n(\ln n)^{-1+\varepsilon_2}), \quad (12)$$

where $\tilde{Q}_j(n)$ is the number of solutions of equation (4) under the additional condition $x < \sqrt{n} n_1^{-1}$, $n_1 = \exp(\ln n)^{\varepsilon_0}$, $\varepsilon_0 > 0$ a small constant. In this case

$$\tilde{Q}_j(n) = \sum_{s=1}^k C_k^s Q_s(n), \quad (13)$$

where $Q_s(n)$ is the number of solutions of equation (4) with $x < \sqrt{n} n_1^{-1}$, when among the numbers $p_1, p_2, \dots, p_k$ there are s numbers $q_1, q_2, \dots, q_s$, each of which is greater than $\exp(\ln n)^{\alpha_0}$, $\alpha_0 > 0$ a sufficiently small constant, while the remaining $p_i \leq$

$$\leq \exp(\ln n)^{\alpha_0}.$$

Thus, $Q_s(n)$ is the number of solutions of the equation

$$p_{i_1} p_{i_2} \dots p_{i_{k-s}} q_1 q_2 \dots q_s + (-1)^{j-1} xy = n_j \quad (j = 1, 2) \quad (14)$$

under the conditions indicated above.

In order to reduce equation (14) to a form convenient for applying the dispersion method, we divide the numbers $q_1 q_2 \dots q_s$ into two classes:

I. $q_1 q_2 \dots q_s$ has at least one divisor q_i such that

$$\exp(\ln n)^{\alpha_0} < q_i \leq P_0(n), \quad (15)$$

where

$$P_0(n) = \exp \ln n \frac{\ln \ln \ln n}{K_0 \ln \ln n},$$

K_0 is a sufficiently large constant.

II. $q_1 q_2 \dots q_s$ has no divisors $q_i \leq P_0(n)$, i.e. all $q_i > P_0(n)$.

Let $q_1 q_2 \dots q_s \in \text{I}$. Taking for v the least of the q_i satisfying (15), we write equation (14) in the form

$$vD' + (-1)^{j-1} xy = n_j \quad (j = 1, 2), \quad (16)$$

where $D' \in (D)$, $v \in (v)$. Equation (16) is solved by the ordinary dispersion method according to the scheme of work ⁽³⁾.

If $q_1 q_2 \dots q_s \in \text{II}$, then equation (14), with the aid of a generalization of the elementary sieve (see ⁽¹⁾, (0, 6, 16)), is reduced to equations of the form

$$p_{i_1} \dots p_{i_{k-s}} x'_1 \dots x'_{k_1} \dots z'_1 \dots z'_{k_s} + (-1)^{j-1} xy = n_j, \quad (17)$$

where $x'_1, \dots, x'_{k_1}, \dots, z'_1, \dots, z'_{k_s}$ have no prime divisors $\leq P_0(n)$. Equation (17) is solved with the aid of the ordinary dispersion method when $k_1 + \dots + k_s \geq 7$, and of the complicated dispersion method when $k_1 + \dots + k_s \leq 7$ (see ⁽³⁾).

2) We represent $S_j(n)$, following C. Hooley ⁽⁹⁾ and B. M. Bredikhin ⁽³⁾, as follows:

$$\begin{aligned}
 S_j(n) = & 4 \sum_{p_1 \dots p_k + (-1)^{j-1} 2^\lambda xy = n_j} \chi_4(x) - 8 \sum_{\substack{p_1 \dots p_k + (-1)^{j-1} 2^\lambda xy = n_j \\ x \leq \sqrt{n} n_1^{-1}}} \chi_4(x) \\
 & - 4 \sum_{\substack{p_1 \dots p_k + (-1)^{j-1} 2^\lambda xy = n_j \\ \sqrt{n} n_1^{-1} < x < \sqrt{nn_1} \\ y < \sqrt{nn_1}}} \chi_4(x) + 4 \sum_{\substack{p_1 \dots p_k + (-1)^{j-1} 2^\lambda xy = n_j \\ \sqrt{n} n_1^{-1} < x < \sqrt{nn_1}}} \chi_4(x) \quad (18) \\
 & + O\left(\frac{n}{(\ln n)^C}\right) = \Sigma_A - \Sigma_{B_1} + \Sigma_{B_2} + O\left(\frac{n}{(\ln n)^C}\right),
 \end{aligned}$$

where C is a large constant. Σ_A is calculated by the dispersion method according to the scheme of the proof of Theorem 1 (with certain complications), while Σ_{B_1} and Σ_{B_2} are estimated by C. Hooley's method ⁽⁹⁾ according to the scheme developed in ⁽³⁾.

3) Equation (10) is first solved for the case when the values of $\varphi(x, y)$ run through a whole sequence of quadratic forms. In doing so, the dispersion method is applied according to the scheme of work ⁽³⁾, taking into account the considerations expressed in solving equation (4). The transition to an individual form is carried out according to the scheme of work ⁽⁴⁾.

Kuibyshev Pedagogical Institute
named after V. V. Kuibyshev

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