

ON THE GREEN' S FUNCTION OF THE STURM-LIOUVILLE EQUATION WITH OPERATOR COEFFICIENTS

È. Abdukadyrov

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Abstract**Full Text**

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MATHEMATICS**E. ABDUKADYROV****ON THE GREEN' S FUNCTION OF THE STURM-LIOUVILLE EQUATION WITH OPERATOR COEFFICIENTS***(Presented by Academician A. N. Tikhonov on 21 IV 1970)*

1. Let H be a separable Hilbert space. Consider the set of all Bochner-measurable functions $f(x)$ ($-\infty < x < \infty$) with values in H , for which

$$\int_{-\infty}^{\infty} \|f(x)\|_H^2 dx < \infty.$$

This set of functions forms a new Hilbert space H_1 (also separable), if the scalar product in it is defined by the equality

$$(f(x), g(x))_{H_1} = \int_{-\infty}^{\infty} (f(x), g(x))_H dx.$$

Let $P(x)$ and $Q(x)$ ($-\infty < x < \infty$) be families of operators in H . In the space H_1 consider the differential expression

$$l(y) = -(P(x)y')' + Q(x)y, \quad (1)$$

defined on some set of functions $D \subset H_1$ (derivatives are understood in the strong sense).

It can be shown that, under certain restrictions on the families of operators $P(x)$ and $Q(x)$, the differential expression l can be extended to a self-adjoint operator $\bar{L}$ in H_1 .

In the work of B. M. Levitan ⁽¹⁾, the operator Green' s function of the operator $\bar{L}$ was studied for $P(x) = E$, where E is the identity operator in the space H . The principal aim of the present note is the study of the operator Green' s function of the operator $\bar{L}$ for $P(x) \neq E$.

2. We list the main restrictions on the operators $P(x)$ and $Q(x)$ under which we have been able to study the operator Green's function of the operator $\bar{L}$:

- 1) For all x ($-\infty < x < \infty$) and all $f \in H$,

$$m(f, f)_H \leq (P(x)f, f)_H \leq M(f, f)_H,$$

where m and M are constant positive numbers.

- 2) The operator function $P(x)$ ($-\infty < x < \infty$) is everywhere uniformly differentiable.
- 3) The operators $Q(x)$ are self-adjoint in H for almost all x , and for all (almost all) x there exists a common everywhere dense in H set $D\{Q(x)\}$, on which the $Q(x)$ are defined and symmetric*.
- 4) The operators $Q(x)$ are uniformly bounded from below; that is, there exists a constant number $d < \infty$ such that for almost all x and all $f \in D\{Q(x)\}$,

$$(Q(x)f, f)_H \geq -d(f, f)_H.$$

- 5) There exist constant numbers $A > 0$, $0 < a < 3/2$ such that for all x and $|x - \xi| \leq 1$ the inequality

$$\| [Q(\xi) - Q(x)]Q^{-a}(x) \|_H < A|x - \xi|$$

holds.

- 6) For $|x - \xi| > 1$,

$$\| K(\xi) \exp\{-\frac{1}{2}|x - \xi|\omega\} \|_H < B,$$

where $\omega = \{K(x) + \mu P^{-1}(x)\}^{1/2}$, $K(x) = P^{-1/2}(x)Q(x)P^{-1/2}(x)$, and B is a constant number.

* Thus, we admit that the operators $Q(x)$ may be unbounded in H .

- 7) For all x and ξ from $(-\infty, \infty)$

$$\| Q(x)P^{\pm 1/2}(x)Q^{-1}(x) \|_H < B_1, \quad \| Q(\xi)P^{-1/2}(x)P^{1/2}(\xi)Q^{-1}(\xi) \|_H < B_2,$$

where B_1, B_2 are constants.

Suppose that $Q(x)$, for almost all x , is the inverse of a completely continuous operator. Then $K(x)$, too, for almost all x , is the inverse of a completely continuous operator. Denote by $\beta_1(x) \leq \beta_2(x) \leq \beta_3(x) \leq \dots$ the eigenvalues

of the operator $K(x)$, with respect to which we shall assume that they are measurable functions. Without loss of generality, one may assume that $\beta_1(x) \geq 1$.

8) The series

$$\sum_{i=1}^{\infty} \beta_i^{-3/2}(x)$$

converges and its sum $F(x)$ is a function of the class

$$L_1(-\infty, \infty), \quad \text{i.e.} \quad \int_{-\infty}^{\infty} F(x) dx < \infty.$$

3. If conditions 1), 3), and 4) are satisfied, there exists a Friedrichs self-adjoint extension of the operator l . We denote it by L . The operator L in H_1 is bounded below by the same number d . Therefore, if $\mu > d$, then there exists the inverse operator $(L + \mu E)^{-1} = R_\mu$. The following holds.

Theorem 1. If conditions 1)–8) are fulfilled, then the operator R_μ , for all $\mu > d$, is an integral operator with operator kernel $G(x, \eta; \mu)$, which is an operator function in H , depending on two variables x and η ($-\infty < x, \eta < \infty$) and on the parameter μ , and satisfies the following conditions:

- a) $G(x, \eta; \mu)$ is strongly continuous in the variables (x, η) ;
- b) there exists a strong derivative $\partial G / \partial \eta$, and

$$\frac{\partial G}{\partial \eta}(x, x + 0; \mu) - \frac{\partial G}{\partial \eta}(x, x - 0; \mu) = -P^{-1}(x),$$

the equality being understood in the sense of equality (0.5) of paper ⁽¹⁾;

c)

$$-(G'_\eta P(\eta))'_\eta + G\{Q(\eta) + \mu E\} = 0,$$

the equality being understood in the sense of equality of item c) of paper ⁽¹⁾;

d) $G^*(x, \eta; \mu) = G(\eta, x; \mu)$.

Here the asterisk denotes the adjoint operator in H .

4. The theorem is proved by the method of paper ⁽¹⁾, using some results from ⁽²⁾ and the following lemmas.

Put

$$g(x, \eta; \mu) = \frac{1}{2} P^{-1/2}(x) \omega^{-1} e^{-|x-\eta|\omega} P^{-1/2}(x)$$

and consider the integral equation

$$\begin{aligned} G(x, \eta; \mu) = & g(x, \eta; \mu) - \int_{-\infty}^{\infty} g(x, \xi; \mu) [Q(\xi) - Q(x)] G(\xi, \eta; \mu) d\xi + \\ & + \int_{-\infty}^{\infty} \left[\frac{1}{2} P^{-1/2}(x) \omega \exp(-|x - \xi|\omega) P^{-1/2}(x) (P(\xi) - P(x)) + g'_\xi(x, \xi; \mu) P'(\xi) \right] \times \\ & \times G(\xi, \eta; \mu) d\xi = g(x, \eta; \mu) - NG. \end{aligned} \quad (2)$$

Consider the Banach spaces $X_1, X_2, X_3^{(p)}$ ($p \geq 1$), $X_1^{(s)}, X_2^{(s)}, X_4^{(s)}$ ($s \leq 0$), whose elements are operator functions $A(x, s)$ in H ($-\infty < x, s < \infty$) (the definitions of these spaces are given in paper ⁽¹⁾).

Lemma 1. If the operator functions $P(x)$ and $Q(x)$ satisfy conditions 1)–7), then for sufficiently large μ ($\mu > \mu_0$) the operator N is a contraction operator in the spaces $X_1, X_2, X_1^{(s)}, X_2^{(s)}$, and $X_4^{(s)}$.

In all the Banach spaces considered above, equation (2) has a unique solution, which can be obtained by iteration if and only if the operator function $g(x, \eta; \mu)$ belongs to the corresponding space. It is easy to verify that $g(x, \eta; \mu) \in X_3 \equiv X_3^{(1)}$. Therefore $G(x, \eta; \mu)$, for sufficiently large μ ($\mu > \mu_0$), also belongs to X_3 , provided only that the operator functions $P(x)$ and $Q(x)$ satisfy conditions 1)–7). If, in addition to these conditions, condition 8) also holds, then $g(x, \eta; \mu) \in X_2$, and, consequently, $G(x, \eta; \mu)$, for sufficiently large μ ($\mu > \mu_0$), is also an element of the space X_2 .

Lemma 2. Let the operator functions $P(x)$ and $Q(x)$ satisfy conditions 1), 3), 4), 6) and 7), and the condition: for $|x - \eta| \leq 1$,

$$\|Q^{-1/2}(x)Q^{1/2}(\eta)\|_H \leq C,$$

where C is a constant.

Then the operator function $g(x, \eta; \mu)$ belongs to the space $X_4^{(1/2)}$, i.e.

$$\sup_{-\infty < x < \infty} \int_{-\infty}^{\infty} \|g(x, \eta; \mu)Q^{1/2}(\eta)\|_H d\eta < \infty.$$

If $Q(x)$ also satisfies conditions 2), 5), then $G(x, \eta; \mu)$, for sufficiently large $\mu > \mu_0$, also belongs to the space $X_4^{(1/2)}$.

Lemma 3. Let the operator functions $P(x)$ and $Q(x)$ satisfy conditions 1), 3), 4), 6) and 7), and the condition: for $|x - \eta| \leq 1$,

$$\|Q^{1/2}(x)Q^{-1/2}(\eta)\|_H \leq C.$$

Then the operator function $\partial^2 g / \partial \eta^2$ ($\eta \neq x$) belongs to $X_4^{(-1/2)}$, i.e.

$$\sup_{-\infty < x < \infty} \int_{-\infty}^{\infty} \left\| \frac{\partial^2 g}{\partial \eta^2} Q^{-1/2}(\eta) \right\|_H d\eta < \infty.$$

Now let $\mu' \in (d, \mu_0]$. Consider the integral equation

$$G(x, \eta; \mu') = G(x, \eta; \mu) + (\mu - \mu') \int_{-\infty}^{\infty} G(x, s; \mu) G(s, \eta; \mu') ds, \quad (3)$$

where $G(x, s; \mu)$ is the Green operator function of the operator L , constructed above, for $\mu > \mu_0$. By the preceding,

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \|G(x, s; \mu)\|_2^2 ds \right\} dx < \infty. \quad (4)$$

Here $\|G(x, s; \mu)\|_2$ is the Hilbert–Schmidt norm of the operator $G(x, s; \mu)$ in H for fixed $(x, s; \mu)$. By virtue of (4) (see ^(1, 3)) the integral operator

$$Af = \int_{-\infty}^{\infty} G(x, s; \mu) f(s) ds, \quad f(s) \in H_1$$

is completely continuous in H_1 . Since $\mu - \mu'$ is a resolvent point of the operator A (see ⁽⁴⁾), the integral equation (3) has a unique solution $G(x, \eta; \mu')$. Splitting the integral operator in (3) into a finite-dimensional operator and an integral operator whose norm is sufficiently small, one can show that the operator $G(x, \eta; \mu')$ is uniformly bounded in H . By virtue of the properties of the Green operator function $G(x, \eta; \mu)$, one can prove that $G(x, \eta; \mu')$ is the (unique) Green operator function of the operator L for $\mu' \in (d, \mu_0)$.

5. Analogous results can be obtained in the case of the half-line $(0, \infty)$. Let the operator functions $P(x)$ and $Q(x)$ be defined on the half-line $(0, \infty)$ and satisfy conditions 1)–8). Consider the operator

$$l(y) = -(P(x)y')' + Q(x)y, \quad (0 < x < \infty) \quad (5)$$

and adjoin to it the boundary condition at zero

$$y'(0) = hy(0), \quad (6)$$

where h is a constant self-adjoint operator in H (possibly unbounded).

Let us show how the analogue of the function $g(x, \eta; \mu)$ is constructed. Put

$$K(x) = P^{-1/2}(x)Q(x)P^{-1/2}(x), \quad \omega = \{K(x) + \mu P^{-1}(x)\}^{1/2},$$

$$X(\eta) = P^{-1/2}(x)[\operatorname{ch} \omega \eta + h\omega^{-1} \operatorname{sh} \omega \eta]P^{-1/2}(x) \quad (0 < x, \eta < \infty).$$

Then

$$g(x, \eta; \mu) = \begin{cases} X(\eta)P^{1/2}(x)(\omega + h)^{-1}e^{-\omega x}P^{-1/2}(x), & \text{if } \eta \leq x, \\ P^{-1/2}(x)e^{-\omega \eta}(\omega + h)^{-1}P^{1/2}(x)X(x), & \text{if } \eta > x. \end{cases}$$

It is easy to verify that the function $g(x, \eta; \mu)$ satisfies the conditions:

$$P(x)\frac{\partial^2 g}{\partial \eta^2} = \{Q(x) + \mu E\}g, \quad \left(\frac{\partial g}{\partial \eta} - hg\right)_{\eta=0} = 0,$$

$$\frac{\partial g}{\partial \eta}\bigg|_{\eta=x+0} - \frac{\partial g}{\partial \eta}\bigg|_{\eta=x-0} = -P^{-1}(x).$$

Having constructed the function $g(x, \eta; \mu)$, one can set up an integral equation analogous to equation (2) (the limits of integration will now be 0 and ∞), and then prove the existence of an operator function $G(x, \eta; \mu)$, which is the kernel of the operator $(L + \mu E)^{-1}$, where L is the Friedrichs extension of the operator (5) + (6).

The operator Green's function on a finite interval is studied analogously. We note that a problem of this type was considered in the works ^(4,5).

6. As an example, consider the eigenvalue problem for an operator of the form

$$-\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(a_i(x) \frac{\partial}{\partial x_i} u(x) \right) + q(x)u(x) = \lambda u(x), \quad (7)$$

$$u|_{\Gamma} = 0 \quad (8)$$

in the cylindrical domain

$$\Pi = \Pi((x_2, x_3) \in \Omega; -\infty < x_1 < \infty), \quad x = (x_1, x_2, x_3),$$

where Ω is a plane simply connected domain of the (x_2, x_3) -plane, and Γ is the boundary of the cylinder Π . The coefficients $a_i(x)$ are positive, uniformly

bounded functions on Π , and $q(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ (by another method, for more general domains, the analogous problem with $a_i(x) \equiv 1$ ($i = 1, 2, 3$) was studied in a paper by A. G. Ramm ⁽⁶⁾). Equality (7) can be rewritten in operator form (in the present case $H = L_2(\Omega)$, $H_1 = L_2(\Omega, -\infty < x_1 < \infty)$)

$$-\frac{d}{dx_1} \left(P(x_1) \frac{du}{dx_1} \right) + \{Q(x_1) - \lambda E\}u = 0,$$

where

$$Q(x_1)u = -\frac{\partial}{\partial x_2} \left(a_2(x) \frac{\partial u}{\partial x_2} \right) - \frac{\partial}{\partial x_3} \left(a_3(x) \frac{\partial u}{\partial x_3} \right) + q(x)u, \quad P(x_1) \equiv a_1(x)E.$$

Applying the results set forth above to this case, one can study the properties of the Green's function of problem (7) + (8) and derive an asymptotic formula for the number of eigenvalues of this problem.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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