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Abstract

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MATHEMATICS

V. I. YUDOVICH

ON THE STABILITY OF FORCED OSCILLATIONS OF A FLUID

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This note gives a justification of the linearization method in the problem of stability of periodic motions of a viscous incompressible fluid. At the same time, a new method of proof is obtained for the case of stationary flows ⁽¹⁾.

1. Problem with initial data. Let the Navier–Stokes equations in a bounded three-dimensional domain Ω with boundary S of class C^2 , under a given external body force and boundary velocity that are T -periodic in time, have a T -periodic-in-time solution* with velocity vector $v_0(x, t)$. Seeking an arbitrary flow v in the form $v = v_0 + u$, we arrive at the nonlinear equation of perturbations

$$\frac{du}{dt} + \nu A_0 u + B(t)u = Ku, \quad (1)$$

where the following notation is used:

$$A_0 u = -\Pi \Delta u; \quad Ku = -K_0(u, u); \quad K_0(u, v) = \Pi(u, \nabla)v,$$

$$K_0^0(u, v) = K_0(u, v) + K_0(v, u); \quad B(t)u = K_0^0(u, v_0).$$

Here Π is the operator of orthogonal projection of the vector space L_2 onto the space $H \subset L_2$, obtained by closing the set of finite smooth solenoidal vectors in Ω .

We shall regard the operators A_0, B, K as defined on the set D_{A_0} of solenoidal vectors of class $W_2^{(2)}$ that vanish on the boundary S . The operator A_0 is self-adjoint and positive definite in H . We denote its energy space by H_1 :

$$(u, v)_{H_1} = (A^{1/2}u, A^{1/2}v)_H. \quad (2)$$

Introduce the set M_T of vector-functions $u(t)$ of time $t \in [0, T]$ with values in H such that $u(t) \in D_{A_0}$ for all $t \in [0, T]$ and the vector-functions $u(t), A_0 u(t)$

have strong derivatives of all orders with respect to t in H . Define the Hilbert space H_2^T as the closure of the set M_T in the metric generated by the scalar product

$$(u, v)_{H_2^T} = \int_0^T [(du/dt, dv/dt)_H + (A_0 u, A_0 v)_H] dt + (u(T), v(T))_{H_1}. \quad (3)$$

Lemma 1. The vector-function $u \in H_2^T$ is strongly continuous in H_1 for $t \in [0, T]$. Moreover, the estimates

$$\|u(t)\|_{H_1} \leq c \|u\|_{H_2^T}; \quad 0 \leq t \leq T; \quad (4)$$

$$\int_0^T \|u(t)\|_{L_p}^{4p/(p-6)} dt \leq c \|u\|_{H_2^T}^{4p/(p-6)}; \quad \int_0^T \|D_{xu}(t)\|_{L_q}^{4q/3(q-2)} dt \leq c \|u\|_{H_2^T}^{4q/3(q-2)}, \quad (5)$$

hold, where c depends only on the domain Ω , $6 \leq p < \infty$, $2 < q \leq 6$.

* The existence theorem for a periodic motion is formulated in (2).

These inequalities are easily derived with the aid of S. L. Sobolev's embedding theorems.

Consider the Cauchy problem for equation (1) with the initial condition

$$u(0) = a, \quad a \in H_1. \quad (6)$$

By its solution on the time interval $[0, T]$ we shall mean a vector function $u \in H_2^T$ satisfying equation (1) for almost all $t \in [0, T]$ and the initial condition (6)—in the sense that

$$\|u(t) - a\|_{H_1} \rightarrow 0 \quad (t \rightarrow 0). \quad (7)$$

Similarly, we define the solution of the Cauchy problem for the linearized equation

$$du/dt + A_0 u + B(t)u = 0. \quad (8)$$

Let $\tilde{H}_2^T$ denote the closure of the set of smooth solenoidal vectors $v(x, t)$ ($x \in \Omega$, $t \in [0, T]$) with respect to the norm

$$\|v\|_{\widetilde{H}_2^T}^2 = \int_0^T \left[\|\partial v / \partial t\|_{L_2(\Omega)}^2 + \|v\|_{W_2^{(2)}(\Omega)}^2 \right] dt. \quad (9)$$

Lemma 2. *Let $v_0 \in \widetilde{H}_2^T$. Then, for any $a \in H_1$, the problems (1, 6) and (8, 6) are uniquely solvable.*

This lemma is not difficult to prove by the method of successive approximations, regarding the terms $B(t)u$ and Ku as perturbations. The necessary estimates follow from Lemma 1 and the identity

$$\|u(t)\|_{H_1}^2 + \int_0^t \left(\left\| \frac{du}{dt} \right\|_H^2 + \|A_0 u\|_H^2 \right) d\tau = \|a\|_{H_1}^2 + \int_0^t \|f(\tau)\|_H^2 d\tau;$$

$$f = \frac{du}{dt} + A_0 u; \quad a = u(0). \quad (10)$$

Introduce the operators N_t , U_t of displacement along the trajectories of equations (1) and (8), putting, for any $a \in H_1$,

$$N_{ta} = u(t); \quad U_{ta} = \tilde{u}(t), \quad (11)$$

where $u, \tilde{u}$ are the solutions of the Cauchy problem (1, 6) and the problem (8, 6).

Lemma 3. *The operator $N_t : H_1 \rightarrow H_1$ ($0 \leq t \leq T$) is defined in a neighborhood of zero in the space H_1 , is completely continuous and continuously differentiable. Its Fréchet differential at the point 0 is U_t . The operator U_t is also completely continuous in H_1 .*

2. Stability condition. We shall say that the solution v_0 is Lyapunov stable in the space H_1 if problem (1, 6), for every a from some neighborhood of zero in the space H_1 , has a solution on the time interval $[0, \infty)$, and for every $\varepsilon > 0$ one can indicate a $\delta > 0$ such that from the condition $\|a\|_{H_1} < \delta$ it follows that $\|N_{ta}\|_{H_1} < \varepsilon$ for $t \geq 0$. If, in addition, $\|N_{ta}\|_{H_1} \rightarrow 0$ ($t \rightarrow \infty$), then we shall say that the flow v_0 is asymptotically stable in H_1 .

Seeking a solution of equation (6) in the form $u(t) = e^{\sigma t} w(t)$, where $w(t)$ is a T -periodic vector function, we arrive at the problem

$$\frac{dw}{dt} + \sigma w + A_0 w + B(t)w = 0; \quad w(t+T) \equiv w(t). \quad (12)$$

The set of complex numbers σ for which problem (12) has a nonzero solution is called the stability spectrum of the flow v_0 and is denoted by $\Sigma(v_0)$.

If $\sigma \in \Sigma(v_0)$, then $\rho = e^{T\sigma}$ is an eigenvalue of the monodromy operator U_T ; conversely, if ρ is an eigenvalue of the operator U_T , then

$$\sigma_k = \frac{1}{T} \ln \rho + \frac{2\pi}{T} K_i \in \Sigma(v_0), \quad k = 0, \pm 1, \dots$$

From the complete continuity of the operator U_T it follows that it can have no more than a finite number of eigenvalues with modulus greater than 1. Therefore the stability spectrum contains no more than a finite number of eigenvalues σ with positive and distinct real parts.

Theorem 1. *Let the stability spectrum of the periodic flow v_0 lie in the left half-plane*

$$\operatorname{Re} \sigma < -\sigma_0 < 0; \quad \sigma \in \Sigma(v_0). \quad (13)$$

Then the flow v_0 is asymptotically stable in H_1 . Moreover, for every solution of the Cauchy problem (1,6), for sufficiently small $\|a\|_{H_1}$, the estimates

$$\|u(t)\| \leq C e^{-\sigma_0 t} \|a\|_{H_1}; \quad \int_0^t e^{2\sigma_0 \tau} \left(\left\| \frac{du}{d\tau} \right\|_H^2 + \|A_0 u\|_H^2 \right) d\tau \leq C^2 \|a\|_{H_1}^2. \quad (14)$$

hold.

The proof of this theorem is obtained by applying to the operator $N_T = N$ the following lemma.

Lemma 4. *Let N be a continuously differentiable operator mapping a neighborhood of zero of the B -space X into X . Let $N(0) = 0$, $N'(0) = U$, and let the spectrum of the operator U be contained inside the unit disk*

$$|\sigma(U)| < \rho < 1. \quad (15)$$

Then $\|N^n x_0\| \rightarrow 0$ ($n \rightarrow \infty$), if $\|x_0\|$ is sufficiently small. Moreover, the estimate

$$\|N^n x_0\| \leq C \rho^n \|x_0\| \quad (16)$$

holds.

3. Condition of instability. Theorem 2. *Let the stability spectrum $\Sigma(v_0)$ contain at least one eigenvalue σ_0 with positive real part. Then the flow v_0 is unstable in H_1 .*

This theorem is derived from the following lemma.

Lemma 5. Let N, U be the same as in Lemma 4. Let the spectrum of the operator U be represented as the union of nonintersecting closed sets $\sigma_1(U)$ and $\sigma_2(U)$, where

$$|\sigma_1(U)| > 1; \quad |\sigma_2(U)| \leq 1. \quad (17)$$

Then there exists $\varepsilon_0 > 0$ such that, for any $\delta > 0$, one can specify a vector $a \in X$ and a natural number n for which

$$\|a\| < \delta; \quad \|N^n a\| \geq \varepsilon_0. \quad (18)$$

4. Conditional stability. Theorem 3. Let $\Sigma(v_0)$ contain no points of the imaginary axis. Then, in a neighborhood of zero of the space H_1 , there are defined a finite-dimensional manifold Y_1 and a manifold of finite codimension Y_2 , possessing the following properties. 1) If $a \in Y_2$, then $N_t a \rightarrow 0$ in H_1 as $t \rightarrow +\infty$. 2) If $\|a\|_{H_1}$ is small and $a \notin Y_2$, then there exists $t > 0$ such that $\|N_t a\|_{H_1} \geq \varepsilon_0$; $\varepsilon_0 > 0$ does not depend on a . 3) If $a \in Y_1$, then the Cauchy problem (1,6) has a solution $N_t a$, defined for all $t < 0$, and $\|N_t a\|_{H_1} \rightarrow 0$ as $t \rightarrow -\infty$. 4) If $a \notin Y_1$ and $\|a\|_{H_1}$ is small, then either $\|N_t a\|_{H_1} \geq 0$ for some $t < 0$, or $N_t a$ is not defined for some $t < t_0 \leq 0$.

This theorem follows from the following lemma.

Lemma 6. Let N, U be the same as in Lemma 4, and suppose the spectrum of the operator U can be represented as the union of closed sets $\sigma_1(U)$ and $\sigma_2(U)$, with

$$|\sigma_1(U)| > 1; \quad |\sigma_2(U)| < 1. \quad (19)$$

Then, in some neighborhood $D_r = \{x \in X : \|x\| < r\}$; $r > 0$ of zero in the space X , there are manifolds Y_1 and Y_2 , invariant with respect to the operator N , which at zero are tangent respectively to the invariant subspaces X_1, X_2 of the operator U corresponding to the spectral sets $\sigma_1(U)$ and $\sigma_2(U)$. Moreover: 1) $\|N^n x\| \rightarrow 0$ if $x \in Y_2$; 2) for every $x \in D_r - Y_2$ there exists a natural number n such that $\|N^n x\| \geq r$; 3) for every $x \in Y_1$ the preimage $N^{-1}x$ is uniquely determined, and $\|N^{-n}x\| \rightarrow 0$ as $n \rightarrow +\infty$; 4) if $x \in D_r - Y_1$, then either, starting from some n , the element $N^{-n}x$ is not defined, or $\|N^{-n}x\| \geq r$ for some n .

The proof of this lemma is close to that given in ⁽³⁾ for the finite-dimensional case.

5. Examples. 1) If the velocity v_0 is small: $\|v_0\|_{L_3(\Omega)} < Cv$ (C is an absolute constant), then the flow v_0 is stable. 2) The conditions of Theorems 1 and 2 are preserved under a small perturbation of the basic flow. Therefore, from each example of a stable or unstable stationary flow ⁽⁴⁻¹¹⁾ one obtains corresponding examples of periodic flows.

Rostov State University

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REFERENCES

- ¹ V. I. Yudovich, DAN, 161, No. 5, 1037 (1965).
- ² V. I. Yudovich, DAN, 130, No. 6, 1214 (1960).
- ³ D. V. Anosov. Scientific Reports of Higher Education Institutions, Phys.-Math. Sciences, No. 1, 3 (1959).
- ⁴ L. D. Meshalkin, Ya. B. Sinai, PMM, 25, 1140 (1961).
- ⁵ A. L. Krylov, DAN, 153, No. 4, 787 (1963).
- ⁶ A. L. Krylov, DAN, 159, No. 5, 978 (1964).
- ⁷ V. I. Yudovich, PMM, 29, No. 3, 453 (1965).
- ⁸ V. I. Yudovich, *Numerical Methods for Solving Problems of Mathematical Physics*, "Nauka," 1966.
- ⁹ V. I. Yudovich, DAN, 169, No. 2, 306 (1966).
- ¹⁰ V. I. Yudovich, PMM, 30, No. 4, 688 (1966).
- ¹¹ V. I. Yudovich, Mat. sbornik, 74 (116), 4, 565 (1967).

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