

ERGODIC THEOREMS FOR MULTICHANNEL SERVICE SYSTEMS

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Abstract

Full Text

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MATHEMATICS

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ERGODIC THEOREMS FOR MULTICHANNEL SERVICE SYSTEMS

1. Let, on the underlying probability space $(\Omega, \mathfrak{F}, P)$, a sequence of pairs of random variables $\{\tau_j^e, \tau_j^s; j \geq 1\}$ be given. We shall say that an m -channel service system with losses is governed by this sequence if calls enter the system at the times $0, \tau_1^e, \tau_1^e + \tau_2^e, \dots$, and the servicing of the j -th call (if such servicing takes place) requires time τ_j^s . If an arriving call finds all m channels busy, then it is refused and drops out of consideration. If, however, the number q_n of busy channels at the moment just before the arrival of the n -th call is less than m , then the call is accepted for service by one of the free channels. Thus, for the systems under consideration one always has $q_n \leq m$. For simplicity we shall assume that $q_1 = 0$ (see Sec. 4).

Along with the "occupancy" q_n , we shall consider more precise characteristics of the systems—the processes $\{q_n(x); n \geq 1, x \geq 0\}$, where the value $q_n(x)$ indicates how many of those calls that were in the system before the moment $t_n = \tau_1^e + \dots + \tau_{n-1}^e$ of arrival of the n -th call will remain in it after time x (i.e., at the moment $t_n + x$), so that $q_n = q_n(0)$.

Suppose that the governing sequence is strictly stationary. Then, without loss of generality, one may assume that on $(\Omega, \mathfrak{F}, P)$ there is given a "two-sided infinite" stationary sequence

$$\{\tau_j^e, \tau_j^s; -\infty < j < \infty\}, \quad (1)$$

by which the system is governed.

2. Let first the number of channels be $m = \infty$. Denote by $I(A)$ the indicator of the event A ,

$$q^k(x) = I(\tau_k^s > \tau_k^e + x) + I(\tau_{k-1}^s > \tau_{k-1}^e + \tau_k^e + x) + \dots + I(\tau_{k-2}^s > \tau_{k-2}^e + \tau_{k-1}^e + \tau_k^e + x) + \dots \quad (2)$$

Theorem 1. *If the sequence (1) is strictly stationary, $\mathbf{M}\tau^s < \infty$, and the sequence $\{\tau_j^e; -\infty < j < \infty\}$ is, moreover, metrically transitive, then the distribution of the processes*

$$\{q_{n,k}(x); k \geq 0; x \geq 0\} = \{q_{n+k}(x); k \geq 0, x \geq 0\}$$

converges monotonically, as $n \rightarrow \infty$, to the distribution of a proper stationary process in k ,

$$\{q^k(x); k \geq 0, x \geq 0\}.$$

More precisely: for every n there exists a process $\tilde{q}^k(x)$ with the same distribution as $q^k(x)$, and such that $q_{n+k}(x) \leq \tilde{q}^k(x)$, it increases with n , and as $n \rightarrow \infty$

$$\mathbf{P} \left(\bigcup_{k=0}^{\infty} \bigcup_{x \geq 0} \{q_{n+k}(x) \neq \tilde{q}^k(x)\} \right) \rightarrow 0.$$

The proof of this theorem is not difficult to obtain, using the stationarity of the governing sequence and the representation

$$\begin{aligned} q_n(x) = & I(\tau_1^s > \tau_1^e + \dots + \tau_{n-1}^e + x) + I(\tau_2^s > \tau_2^e + \dots + \tau_{n-1}^e + x) + \dots \\ & \dots + I(\tau_{n-1}^s > \tau_{n-1}^e + x). \end{aligned}$$

Theorem 2. *If $\{\tau_j^e\}$ and $\{\tau_j^s\}$ are two independent sequences of independent identically distributed random variables and $\mathbf{M}\tau^e < \infty$, then the condition $\mathbf{M}\tau^s < \infty$ is necessary and sufficient for the finiteness of $q_n(x)$. The probabilities*

$$P_k(x) = \mathbf{P}(q^0(x) = k), \quad k = 0, 1, \dots,$$

satisfy the equations

$$\begin{aligned} P_0(x) &= \int_0^\infty d\mathbf{P}(\tau^e < t) \mathbf{P}(\tau^s < t + x) P_0(t + x), \\ P_k(x) &= \int_0^\infty d\mathbf{P}(\tau^e < t) \mathbf{P}(\tau^e \geq t + x) P_{k-1}(x) + \\ &+ \int_0^\infty d\mathbf{P}(\tau^e < t) \mathbf{P}(\tau^s < t + x) P_k(t + x), \quad k = 1, 2, \dots, \end{aligned}$$

which, in the class of systems of functions of bounded variation possessing the properties $P_0(x) \rightarrow 1$, $P_k(x) \rightarrow \infty$, for $k \geq 1$, $x \rightarrow \infty$, have a unique solution.

Hence it is easy to obtain, in particular, that if $\mathbf{P}(\tau^e \geq x) = e^{-\alpha x}$, $\alpha > 0$, then

$$P_k(x) = \frac{1}{k!} \left(\int_x^\infty \alpha \mathbf{P}(\tau^s \geq t) dt \right)^k \exp \left\{ - \int_x^\infty \alpha \mathbf{P}(\tau^s \geq t) dt \right\}.$$

The “occupancy” $q(t, x)$ at time t , i.e., the number of calls that are in the system at time $t + x$ and whose service has already lasted more than x , has analogous properties. If $\mathbf{P}(\tau^e \geq x) = e^{-\alpha x}$, then the limiting distributions of $q_n(x)$ and $q(t, x)$, as $n \rightarrow \infty$, $t \rightarrow \infty$, coincide.

3. The number of channels $m < \infty$. Keeping the notation of the governing sequence and $q_n(x)$, we shall now denote the right-hand side in (2) by $Q_k(x)$. This is, as we have seen, a stationary characteristic of the system governed by the same sequence (1), but with an infinite number of channels. Next denote by A_k the event consisting in the fact that, for some $0 \leq L \leq m - 1$, $l_j \geq 1$, $j = 0, L$, such that $\sum_{j=0}^L l_j = m$, the inequalities

$$\begin{aligned} Q_k(0) &\leq m - l_0, & Q_k(\tau_{k+1}^e) &\leq m - l_0 - l_1, \dots, & Q_k(\tau_{k+1}^e + \dots + \tau_{k+L}^e) &\leq \\ & & & \leq m - l_0 - \dots - l_L = 0 \end{aligned}$$

are satisfied.

Theorem 3. Let the sequence (1) be strictly stationary and metrically transitive. Suppose, in addition, that the condition

$$\mathbf{P}(A_0) > 0. \tag{3}$$

is fulfilled. Then the distribution of the processes $\{q_{n+k}(x); k \geq 0, x \geq 0\}$, as $n \rightarrow \infty$, converges to the distribution of some process stationary in k , $\{q^k(x); k \geq 0, x \geq 0\}$, such that

$$\mathbf{P}(q^0(0) < m) < 1.$$

Convergence here is understood in the same strong sense as in Theorem 1.

There exist various simple sufficient conditions ensuring the fulfillment of (3).

Consider, for example, the case when $\{\tau_j^e\}$, $\{\tau_j^s\}$ are composed of independent random variables. Then for (3) to hold it is sufficient that

$$\mathbf{P}(\tau^s \leq m\tau^e) > 0, \quad \mathbf{M}\tau^s < \infty. \quad (4)$$

A condition in a certain sense opposite will also be sufficient: for all $x \geq x_0$, $\Delta > 0$,

$$\mathbf{P}(\tau^s \in (x, x + \Delta)) > 0, \quad \mathbf{M}\tau^s < \infty \quad (5)$$

for some $x_0 > 0$.

We now describe, for independent τ_j^e, τ_j^s , the limiting distribution $q_n(x)$. The trajectory of the nondecreasing step process $q^0(x)$ is, evidently, completely described by the positions of its jumps. Denote by

$$\mu(\underbrace{0, \dots, 0}_{j \text{ times}}, dx_{j+1}, \dots, dx_m)$$

the probability that $q^0(0) = m - j$, and that the $(m - j)$ jumps of the process $q^0(x)$ are contained respectively in the intervals $dx_k = (x_k, x_k + dx_k)$, $k = j+1, \dots, m$ (the use of the symbol dx_k simultaneously to denote a scalar quantity and an interval will not lead to misunderstandings).

Theorem 4. *If (4) or (5) holds, the stationary distribution $\mu(0, \dots, 0, dx_{j+1}, \dots, dx_m)$, $j = 0, \dots, m$, satisfies the equation*

$$\begin{aligned} \mu(0, \dots, 0, dx_{j+1}, \dots, dx_m) = & \sum_{k=j+1}^m \int_0^\infty \mathbf{P}(\tau^e \in dt) \mathbf{P}(\tau^s \in t + dx_k) M_{1,k} \\ & + \int_0^\infty \mathbf{P}(\tau^e \in dt) \mathbf{P}(\tau^s \leq t) M_2 + \int_0^\infty \mathbf{P}(\tau^e \in dt) M_3, \end{aligned} \quad (6)$$

where

$$\begin{aligned} M_{1,k} = & \int_{0 \leq y_2 \leq y_3 \leq \dots \leq y_{j+1} \leq t} \dots \int \mu(0, dy_2, \dots, dy_{j+1}, dx_{j+1} + t, \dots, dx_{k-1} + t, \\ & + t, dx_{k+1} + t, \dots, dx_m + t). \end{aligned}$$

$$M_2 = \int_{0 \leq y_2 \leq y_3 \leq \dots \leq y_j \leq t} \dots \int \mu(0, dy_2, \dots, dy_j, dx_{j+1} + t, \dots, dx_m + t),$$

$$M_3 = \int_{0 < y_1 \leq y_2 \leq \dots \leq y_j \leq t} \dots \int \mu(dy_1, \dots, dy_j, dx_{j+1} + t, \dots, dx_m + t).$$

This equation has a unique solution satisfying the condition

$$\int_{0 \leq x_1 \leq \dots \leq x_m} \dots \int \mu(dx_1, \dots, dx_m) = 1.$$

In these formulas, by the interval dy when $y = 0$ one should understand the point 0. Thus the integrals $M_{1,k}$ and M_2 also include “discrete” values, such as $\mu(0, \dots, 0, dx_{j+1} + t, \dots, dx_m + t)$ (in M_2) and others. They do not enter into M_3 .

In the equation for $\mu(0, \dots, 0)$, the terms containing $M_{1,k}$ will be absent on the right-hand side; in the equation for $\mu(dx_1, \dots, dx_m)$ with $x_1 > 0, \dots, x_m > 0$, the term containing M_2 will be absent. The integral M_3 in this equation becomes $\mu(dx_1 + t, \dots, dx_m + t)$.

By direct verification one can establish, for example, that in the case

$$\mathbf{P}(\tau^e \geq x) = e^{-\alpha x}, \quad M\tau^s = a < \infty$$

$$\mu(0, \dots, 0, dx_{j+1}, \dots, dx_m) = ca^{m-j} \mathbf{P}(\tau^s \geq x_{j+1}) \dots \mathbf{P}(\tau^s \geq x_m) dx_{j+1} \dots dx_m,$$

$$c = \left[\sum_{k=0}^m \frac{(a\alpha)^k}{k!} \right]^{-1},$$

satisfies equation (6) and, consequently, is the stationary distribution $q^k(x)$.

As in the case $m = \infty$, here one can also establish that, for the exponential distribution of τ^e , the limiting distributions $q_n(x)$ and $q(t, x)$ as $n \rightarrow \infty, t \rightarrow \infty$ coincide. From this, in particular, will follow Sevast'yanov's theorem concerning Erlang's formulas for the limiting, as $t \rightarrow \infty$, distribution $q(t, x)$.

When τ_j^e, τ_j^s are independent, one can also estimate the rate of convergence of the distribution $q_n(x)$ to the stationary one. If, for example, $\mathbf{P}(\tau^s \leq \tau^e) > 0$ and $Me^{\lambda\tau^s} < \infty$ for some $\lambda > 0$, then this rate of convergence is related to an exponential one.

For $m = 1$, Theorem 3 implies

Theorem 5. *If the sequence (1) is strictly stationary and metrically transitive and*

$$\mathbf{P}(\tau_0^s \leq \tau_0^e, \tau_{-1}^s \leq \tau_{-1}^e + \tau_0^e, \tau_{-2}^s \leq \tau_{-2}^e + \tau_{-1}^e + \tau_0^e, \dots) > 0,$$

then there exists

$$p(x) = \lim_{n \rightarrow \infty} \mathbf{P}(q_n(x) = 1). \quad (7)$$

If the vectors (τ_i^e, τ_i^s) are independent and d is the greatest common divisor of the numbers k for which

$$r_k = \mathbf{P}(\tau_1^s \in (X_{k-1}, X_k)) > 0, \quad X_k = \sum_{j=1}^k \tau_j^e,$$

then, for the existence of (7), it is necessary and sufficient that $d = 1$. If this condition is fulfilled,

$$p(x) = \left(\sum_{k=1}^{\infty} k r_k \right)^{-1} \sum_{j=1}^{\infty} \mathbf{P}(\tau_1^s > X_j + x).$$

4. The theorems on the existence of a stationary limiting distribution for the sequence $\{q_{n+k}(x); k \geq 0, x \geq 0\}$ as $n \rightarrow \infty$, formulated above, admit generalization in the following three directions simultaneously:

- 1) For arbitrary proper initial conditions (at time 0, q_0 channels are busy and the service times of the "initial" calls are equal to $\rho_1, \dots, \rho_{q_0}$).
- 2) Calls may arrive in batches. In this case, on $(\Omega, \mathfrak{F}, P)$ one should consider strictly stationary sequences, say, of the form $\{\tau_j^e, \nu_j^e, \tau^s; -\infty < j < \infty\}$, where ν_j^e is the number of calls arriving in the j -th batch, $\tau^s = (\tau_{j,1}^s, \dots, \tau_{j,\nu_j^e}^s)$ is the vector of service times of the calls in the j -th batch. In this case the condition $M\tau^s < \infty$ in Theorem 1 must be replaced by the requirement $M[\tau^s] < \infty$, where $[\mathbf{x}]$ denotes the sum of the coordinates of the vector $\mathbf{x}$.
- 3) For the existence of the limiting distribution $q_{n+k}(x)$, the sequence (1) need not be stationary at all. It is sufficient only that, in a certain sense, the sequences $\{\tau_j^e, \tau_j^s; j \geq 0\}$ converge as $n \rightarrow \infty$ to a stationary one.

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