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Abstract

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MATHEMATICS

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ON THE OBSERVABILITY OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUA- TIONS IN A HILBERT SPACE

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The problem of observability has by now been investigated by various methods, and many interesting results have been obtained in it (¹, ²). In the present paper a new approach to the study of this problem is proposed, making it possible, for a fairly broad class of nonlinear differential equations, to indicate sufficient conditions for observability.

Let E be a Hilbert space and let R be a normed ring of linear bounded operators mapping E into itself. Consider the differential equation

$$dx/dt = A(t)x + \varphi(x) + p(t), \quad x(t_0) = x_0, \quad (1)$$

where $x = x(t)$, $t_0 \leq t < \infty$, is the unknown continuous, continuously differentiable vector-valued function with values in E ; $A(t)$, $t_0 \leq t < \infty$, is a uniformly bounded and continuous operator-valued function in the sense of the operator norm with values in R ; $\varphi(x)$ is a nonlinear vector-valued function with values in E , continuous in E and, for all x and ξ from E , satisfying in E the Lipschitz condition with constant q :

$$\|\varphi(\xi) - \varphi(x)\| \leq q\|\xi - x\|; \quad (2)$$

$p(t)$, $t_0 \leq t < \infty$, is a bounded and continuous vector-valued function with values in E ; x_0 is some constant vector from $S_\rho(\xi_0)$, where $S_\rho(\xi_0)$ is the ball in E of radius ρ with center at the point ξ_0 . The vector x_0 is assumed to be unknown.

Introduce for consideration the vector-valued function $y(t)$, $t_0 \leq t < \infty$, with values in E :

$$y(t) = C(t)x(t), \quad (3)$$

which we shall call the **trace** of the solution $x(t)$ of equation (1) with initial condition x_0 . Here $C(t)$, $t_0 \leq t < \infty$, is a uniformly bounded and continuous operator-valued function from R ; $x(t)$ is the solution of equation (1), unknown to us, with initial condition x_0 . The existence of the operator-valued function $C^{-1}(t)$, $t_0 \leq t < \infty$, is not assumed.

For equation (1) and the vector-valued function (3), consider the observability problem.

Definition 1. If, from the trace $y(t)$ of the solution $x(t)$ known on the interval $[t_0, t_1]$, one can determine $x(t_1)$, then we shall say that the solution $x(t)$ of equation (1) is **observable**.

In the case when E is an n -dimensional Euclidean space and system (1) is linear, the problem of finding $x(t_1)$ from a linear combination of the components of the vector $x(t)$ known on the interval $[t_0, t_1]$ was considered by R. Kalman ⁽¹⁾ and was called by him the observability problem.

Definition 2. If, from the trace $y(t)$ of the solution $x(t)$ of equation (1) known on the interval $[t_0, t_1]$, one can find a continuous, continuously dif-

ferentiable vector function $\xi(t)$, $t_0 \leq t < \infty$, with values in E , such that, for a prescribed scalar quantity $\mu > 0$, at the instant $t = t_1$ the relation

$$\|\xi(t_1) - x(t_1)\| \leq \mu \quad (4)$$

is satisfied.

Then we shall say that the solution $x(t)$ of equation (1) is μ -observable at the instant $t = t_1$.

Definition 3. Suppose that, from the known trace $y(t)$, $t_0 \leq t < \infty$, of the solution $x(t)$ of equation (1), one can find a continuous, continuously differentiable vector function $\xi(t)$, $t_0 \leq t < \infty$, with values in E , such that, for a prescribed scalar quantity $\mu > 0$, starting from some instant $T(\mu) \in (t_0, \infty)$, for all $t \geq T(\mu)$ the relation

$$\|\xi(t) - x(t)\| \leq \mu \quad (5)$$

is satisfied.

Then we shall say that the solution $x(t)$ of equation (1) is asymptotically μ -observable.

It is obvious that if $t_1 \geq T(\mu)$, then inequality (4) follows from inequality (5), and, consequently, asymptotic μ -observability of a solution implies its μ -observability at the instant $t = t_1$.

The vector function $\xi(t)$, $t_0 \leq t < \infty$, will be called a function that realizes observation.

In this paper we shall find conditions sufficient for μ -observability at the instant $t = t_1$ and for asymptotic μ -observability of the solution $x(t)$, and we shall obtain a differential equation whose solution is a function that realizes observation.

Consider the following auxiliary differential equation

$$d\xi/dt = A(t)\xi + \varphi(\xi) + p(t) + u(t), \quad \xi(t_0) = \xi_0, \quad (6)$$

whose solution $\xi(t)$ is assumed known to us. Here $\xi(t)$, $t_0 \leq t < \infty$, is a continuous, continuously differentiable vector function with values in E ; $u(t)$, $t_0 \leq t < \infty$, is some vector function with values in E . The trace of the solution of equation (6) with initial condition ξ_0 will be denoted by $\eta(t)$:

$$\eta(t) = C(t)\xi(t). \quad (7)$$

In the space E consider the vector function

$$z = \xi - x. \quad (8)$$

It follows from (1) and (6) that $z(t)$ satisfies the differential equation

$$dz/dt = A(t)z + \varphi(\xi) - \varphi(x) + u(t) \quad (9)$$

with initial condition $z(t_0) = z_0$, where $\|z(t_0)\| \leq \rho$. Denote $\varphi(\xi) - \varphi(x) = F(z, t)$; then equation (9) can be written in the form

$$dz/dt = A(t)z + u(t) + F(z, t), \quad (10)$$

where, as follows from (2), $\|F(z, t)\| \leq q\|z\|$. The corresponding equation of first approximation for equation (10) has the form

$$dz/dt = A(t)z + u(t). \quad (11)$$

We take the vector function $u(t)$ to be the following:

$$u(t) = B(t)(\eta(t) - y(t)), \quad (12)$$

where $B(t)$, $t_0 \leq t < \infty$, is some continuous uniformly bounded operator from R . In accordance with (3), (7), (8), and (12), equation (10) takes the form

$$dz/dt = (A(t) + B(t)C(t))z + F(z, t). \quad (13)$$

Equation (11) is written in the form

$$dz/dt = (A(t) + B(t)C(t))z. \quad (14)$$

Sufficient conditions for the asymptotic μ -observability of the solution $x(t)$ of equation (1) are formulated in the form of the following theorem:

Theorem 1. *For the asymptotic μ -observability of the solution $x(t)$ of equation (1) in the space E , it is sufficient to choose an operator $B(t) = B_V(t) \subset R$ so that there exists a differentiable uniformly positive operator-function $V(t)$*

$$0 < a_1(z, z) \leq (V(t)z, z) \leq a_2(z, z), \quad (15)$$

possessing the property that, if $z(t)$ is a solution of equation (14), then

$$\frac{d}{dt}(V(t)z(t), z(t)) \leq -\beta(z(t), z(t)) \quad (16)$$

and the relation $q_1 = v_0/N_0 > q$ holds, where $v_0 = \beta/2a_2$, $N_0 = \sqrt{a_2/a_1}$.

To prove Theorem 1, we note that, when conditions (15) and (16) are fulfilled, for the norm of the solution $z(t)$ of equation (13) the estimate ⁽³⁾

$$\|z(t)\| \leq N_0 e^{-(v_0 - N_0 q)(t - t_0)} \|z(t_0)\|$$

holds for all $\|z(t_0)\| < \rho$.

Then, beginning from the time $T(\mu)$,

$$T(\mu) = [\ln \mu - \ln N_0 - \ln \rho - t_0(v_0 - N_0 q)] / (N_0 q - v_0), \quad (17)$$

for all $t \geq T(\mu)$ inequality (5) will hold, which means asymptotic μ -observability of the solution $x(t)$ of equation (1).

When the conditions of Theorem 1 are fulfilled, there is a theorem that makes it possible to find the function carrying out the observation.

Theorem 2. *The vector-function $\xi(t)$ carrying out asymptotic μ -observation of the solution $x(t)$ of equation (1) is a solution of the differential equation*

$$d\xi/dt = A(t)\xi + \varphi(\xi) + p(t) + B_V(t)C(t)\xi - B_V(t)y(t), \quad \xi(t_0) = \xi_0.$$

In relation (17), μ , q , and ρ are prescribed scalar quantities; $N_0 = N_0(a_1, a_2)$, $v_0 = v_0(a_2, \beta)$. Obviously, by choosing a_1 , a_2 , and β , one can ensure that the relation

$$T(\mu) \leq t_1. \quad (18)$$

is fulfilled.

The scalar quantities a_1 , a_2 , and β for which relation (18) is fulfilled will be denoted by a_1^1 , a_2^1 , and β^1 , respectively. There is a theorem giving sufficient conditions for the μ -observability of the solution $x(t)$ of equation (1) at the time $t = t_1$:

Theorem 3. *For the μ -observability of the solution $x(t)$ of equation (1) at the time $t = t_1$, it is sufficient to choose an operator $B(t) = B_{V_1}(t) \subset R$ so that there exists a bounded operator-function V_1 satisfying relations (15) and (16) for $a_1 = a_1^1$, $a_2 = a_2^1$, and $\beta = \beta^1$.*

When the conditions of Theorem 3 are fulfilled, there is a theorem analogous to Theorem 2:

Theorem 4. *The vector-function $\xi(t)$ carrying out μ -observation of the solution $x(t)$ of equation (1) at the time $t = t_1$ is a solution of the differential equation*

$$d\xi/dt = A(t)\xi + \varphi(\xi) + p(t) + B_{V_1}(t)C(t)\xi - B_{V_1}(t)y, \quad \xi(t_0) = \xi_0.$$

Corollary 1. *If in the right-hand side of equation (1) there is also an unknown perturbation to us—a vector-function $p(t)$, $t_0 \leq t < \infty$, with values in E , such that $\|p(t)\| < k_1$ for all $t_0 \leq t < \infty$, then the equation*

(13) takes the form

$$dz/dt = (A(t) + B_\nu(t)C(t))z + F(z, t) - p_1(t). \quad (19)$$

For the solution $z(t)$ of equation (19), obviously, the estimate holds

$$\|z(t)\| \leq N_0 e^{-(\nu_0 - N_0 q)(t-t_0)} \|z(t_0)\| + \frac{N_0 k_1}{\nu_0} [1 - e^{-\nu_0(t-t_0)}],$$

from which it follows that one can solve the problem of μ -observability under constantly acting perturbations.

Corollary 2. Suppose that relation (3) for the trace $y(t)$ of a solution $x(t)$ of equation (1) has the form

$$y(t) = C(t)x(t) + p_2(t),$$

where $p_2(t)$, $t_0 \leq t < \infty$, is an unknown vector-function with values in E , whose norm $\|p_2(t)\| < k_2$ for all $t_0 \leq t < \infty$. Then equation (13) takes the form

$$dz/dt = (A(t) + B_\nu(t)C(t))z + F(z, t) + p_1(t),$$

where $p_1(t) = B_\nu(t)p_2(t)$ and $\|p_1(t)\| \leq \|B_\nu(t)\| \|p_2(t)\| < b_\nu k_2$, since for all $t_0 \leq t < \infty$, by virtue of the uniform boundedness of the operator-function B_ν , the relation $\|B_\nu(t)\| < b_\nu$ holds. Thus we arrive at the conditions of Corollary 1.

Corollary 3. Suppose that relation (3) for the trace $y(t)$ has the form

$$y(t) = C(t)[x(t) + p_3(t)], \quad (20)$$

where $p_3(t)$, $t_0 \leq t < \infty$, is an unknown vector-function with values in E , $\|p_3(t)\| < k_3$ for all $t_0 \leq t < \infty$. Then, denoting $p_2(t) = C(t)p_3(t)$ and using the fact that $C(t)$ is a uniformly bounded operator-function: $\|C(t)\| < c$ for all $t_0 \leq t < \infty$, we obtain for the norm of $p_2(t)$ the estimate $\|p_2(t)\| = \|C(t)p_3(t)\| < ck_3 = k_2$. Relation (20) takes the form

$$y(t) = C(t)x(t) + p_2(t),$$

and thus we arrive at the conditions of Corollary 2.

Until now, in considering the problem of μ -observability, we have not assumed that the solution $x(t)$ of equation (1) is bounded on the half-interval $t_0 \leq t < \infty$. If we now assume that $\|x(t)\| < k_4$ for all $t_0 \leq t < \infty$, then the following holds.

Corollary 4. Suppose that relation (3) for the trace $y(t)$ of a solution $x(t)$ of equation (1) has the form

$$y(t) = [C(t) + \Phi(t)]x(t), \quad (21)$$

where $C(t)$ is a known operator-function, and $\Phi(t)$, $t_0 \leq t < \infty$, is an unknown uniformly bounded operator-function, $\|\Phi(t)\| < f$. Expression (21) can be written in the form

$$y(t) = C(t)x(t) + p_2(t),$$

where $\|p_2(t)\| = \|\Phi(t)x(t)\| \leq \|\Phi(t)\| \|x(t)\| < fk_4 = k_2$. Thus we find ourselves under the conditions of Corollary 2.

For the finite-dimensional case, sufficient conditions for μ -observability were obtained in the author's work ⁽⁴⁾. We also note that the method developed

above is applicable to finding sufficient conditions for μ -observability of solutions of nonlinear differential equations with retarded argument.

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