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Abstract

Full Text

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MATHEMATICS

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A QUASIANALYTICITY CRITERION FOR ABSTRACT OPERATORS

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The present note is an attempt to extend Carleman's quasianalyticity criterion ^(1,2) to linear operators in a Banach space. We shall be concerned with conditions under which from inequalities of the form

$$\|A^n x\| \leq m_n \|x\| \quad (n = 0, 1, 2, \dots) \quad (1)$$

it follows that $x = 0$. Classical quasianalyticity corresponds to the case when $A = \frac{1}{i} \frac{d}{ds}$ in the space $C[0, 1]$ with the boundary condition $x(0) = 0$. For more general differential operators the problem of quasianalyticity was studied in ⁽³⁻⁵⁾.

The differentiation operator mentioned above is unbounded and has no spectrum. These properties should be retained in the general formulation of the problem: if on some nonzero invariant subspace E the operator A is bounded, then $\|A^n x\| \leq C^n \|x\|$ ($x \in E$, $C = \|A|_E\|$); thus, even with $m_n = C^n$, inequality (1) will be satisfied by a vector $x \neq 0$.

If the operator A has no spectrum, then its resolvent $R(\lambda)$ is an entire function, and it is natural to relate the properties of the operator to the growth of $R(\lambda)$ as $|\lambda| \rightarrow \infty$. For the differentiation operator the function $R(\lambda)$ is of exponential type, bounded on the imaginary axis. We shall take this observation as our starting point.

Let A be an operator without spectrum. The Taylor series of its resolvent has the form

$$R(\lambda) = \sum_{k=0}^{\infty} A^{-k-1} \lambda^k.$$

Put $M_A(r) = \max_{|\lambda|=r} \|R(\lambda)\|$ ($r > 0$). The function $M_A(r)$ as $r \rightarrow \infty$ grows faster than any power of r . Introduce the usual growth characteristic of the sequence $\{m_n\}$:

$$T(r) = \sup_n \frac{r^{n-1}}{m_n}.$$

Lemma 1. *If there exists a vector $x \neq 0$ for which inequalities (1) hold, then*

$$T(r) \leq M_A(r). \quad (2)$$

Proof. We take $\|x\| = 1$. Then

$$1 = \|A^{-n}A^n x\| \leq \|A^{-n}\|m_n \leq M_A(r)r^{-n+1}m_n \quad (3)$$

by Cauchy's inequality. Inequality (2) follows from (3).

In view of Lemma 1, from inequalities (1) it will follow that $x = 0$ whenever the function $T(r)$, in some scale, grows faster than $M_A(r)$. For example:

Theorem 1. If for some $\rho > 0$

$$\int_1^\infty \frac{\ln T(r)}{r^{1+\rho}} dr = \infty, \quad (4)$$

while

$$\int_1^\infty \frac{\ln M_A(r)}{r^{1+\rho}} dr < \infty, \quad (5)$$

then from inequality (1) it follows that $x = 0$.

Condition (4) means that the sequence $\{m_n\}$ satisfies the classical criterion of quasianalyticity "of order ρ " (see (6), pp. 55-56). Theorem 1 is sharp in the following sense.

Theorem 2. Let $\psi(t)$ be a positive logarithmically convex function. Suppose that for some $\rho > 0$

$$\int_1^\infty \frac{\ln \psi(t)}{t^{1+\rho}} dt = \infty. \quad (6)$$

Then there exists an operator A in a Hilbert space, having no spectrum, and such that: 1) $\|R(\lambda)\| \leq \psi(|\lambda|)$; 2) for some $x \neq 0$ the sequence $m_n = \|A^n x\|$ ($n = 0, 1, 2, \dots$) satisfies the quasianalyticity condition (4).

The proof is based on the existence of sequences $\{a_n\}_0^\infty$ and $\{m_n\}_{-\infty}^\infty$ of positive numbers for which

$$\sum_{n=0}^{\infty} a_n^{-1} t^n \leq \psi(t) \quad (t \geq 0),$$

$$a_k m_n \leq m_{k+n+1} \quad (n = 0, \pm 1, \pm 2, \dots; k = 0, 1, 2, \dots)$$

and $\{m_n\}_{n=0}^{\infty}$ satisfies condition (4). In the Hilbert space of sequences $\xi = \{\xi_n\}_{n=-\infty}^{\infty}$ with norm

$$\|\xi\| = \left(\sum_{n=-\infty}^{\infty} |\xi_n|^2 m_n^2 \right)^{1/2}$$

the right-shift operator $A\xi = \{\xi_{n-1}\}_{n=-\infty}^{\infty}$ possesses all the required properties. As x one should take the coordinate unit vector $\{\delta_{n0}\}$.

We now formulate the main criterion of quasianalyticity for order $\rho \leq 1$.

Theorem 3. Let the operator A have no spectrum and

$$\sigma = \overline{\lim}_{r \rightarrow \infty} r^{-\rho} \ln M_A(r) < \infty \quad (0 < \rho \leq 1).$$

Suppose further that on two rays $\arg \lambda = \theta_1$, $\arg \lambda = \theta_2$, $\theta_2 - \theta_1 \equiv \pi \rho^{-1} \pmod{2\pi}$, the estimate $\|R(\lambda)\| \leq \psi(|\lambda|)$ holds, where $\psi(t)$ ($t \geq 0$) is a positive continuous nondecreasing function for which

$$\int_1^{\infty} \frac{\ln \psi(t)}{t^{1+\rho}} \ln t \, dt < \infty. \quad (7)$$

Then, in order that $x = 0$ follow from inequality (1), it is sufficient, and for $\sigma > 0$ necessary, that the quasianalyticity condition (4) be fulfilled.

According to Theorem 2, the quasianalyticity criterion (4) is lost when the condition

$$\int_1^{\infty} \frac{\ln \psi(t)}{t^{1+\rho}} \, dt < \infty, \quad (8)$$

weaker than (7). However, we do not know whether condition (7) in Theorem 3 can be replaced by condition (8).

Let us also note that condition (4) in the case $\sigma = 0$ is not necessary. The proof of Theorem 3 uses the fact (see (7)) that, for $\rho = 1$, $\arg \lambda = 0$, the growth of the resolvent on the real axis can be absorbed by multiplication by a function of the form

$$\varphi(\lambda) = \int_0^a e^{i\lambda t} g(t) dt \quad (a > 0).$$

Owing to this, one can form functions of A of the form

$$\tilde{\omega}(t; A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \varphi(\lambda) R(\lambda) \omega(\lambda) d\lambda,$$

where $\omega(\lambda)$ ($-\infty < \lambda < \infty$) may have arbitrary power growth.

In proving sufficiency (in the case $\rho = 1$) we take $\omega(\lambda) = 1$ and establish that, by virtue of (1),

$$\left\| \frac{d^n}{dt^n} \tilde{\omega}(t; A) x \right\| \leq \text{const} \cdot m_n \|x\|.$$

Then Carleman's theorem implies $\tilde{\omega}(t; A)x = 0$ ($t < 0$). Reversing (9), we obtain the representation

$$R(\lambda)x = \frac{1}{\varphi(\lambda)} \int_0^b e^{i\lambda t} \tilde{\omega}(t; A)x dt,$$

where $b > 0$. By the theorem on addition of indicators ([8], p. 208), the function $R(\lambda)x$ in the upper half-plane has zero exponential type. An analogous conclusion can be obtained for the lower half-plane. After this, owing to condition (7), the matter reduces to Theorem 1.

In proving necessity, the role of $\omega(\lambda)$ is played by the Fourier transform of a finite function from the corresponding non-quasianalytic class. The required vector $x \neq 0$, for which the estimates (1) hold, is obtained in the form $\tilde{\omega}(t; A)y$ for some y and some t .

If the operator A has spectrum, the problem of quasianalyticity disappears, but the problem of completeness arises.

Theorem 4. *Let the resolvent $R(\lambda)$ be meromorphic. Put*

$$m_A(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln \|R(re^{i\theta})\| d\theta, \quad \sigma = \overline{\lim}_{r \rightarrow \infty} r^{-\rho} m_A(r) \quad (0 < \rho \leq 1).$$

Suppose that $\sigma < \infty$ and that, on the two rays $\arg \lambda = \theta_1$, $\arg \lambda = \theta_2$, the conditions of Theorem 3 are satisfied. Suppose, further, that the quasianalyticity condition (4) is satisfied. Then the linear span of the root vectors of the operator A is dense in the set of vectors satisfying inequalities (1).

This theorem is proved by the same method as Theorem 3. We emphasize that the case $\rho > 1$ beyond the scope of Theorems 1, 2 remains uninvestigated.

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Note: Figure translations are in progress. See original paper for figures.

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