

BOUNDARY PROPERTIES OF $\backslash(n\backslash)$ -DIMENSIONAL QUASICONFORMAL MAPPINGS

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Abstract

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MATHEMATICS

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BOUNDARY PROPERTIES OF n -DIMENSIONAL QUASICONFORMAL MAPPINGS

(Presented by Academician M. A. Lavrent'ev on 19 I 1970)

1. Let R^n be n -dimensional Euclidean space; $\widetilde{R}^n = R^n \cup \{\infty\}$ the corresponding Möbius space; $x = (x_1, \dots, x_n)$ a point (or vector) in R^n ; $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ the length of a vector. For an arbitrary domain $\Omega \subseteq R^n$: $L_n(\Omega)$ is the class of measurable functions summable over Ω with exponent n ; $W_n^1(\Omega)$ is the set of functions $\varphi(x) \in L_n(\Omega)$ having generalized derivatives $\partial\varphi/\partial x_i \in L_n(\Omega)$ ($i = 1, \dots, n$); $L_1^n(R^n)$ is the set of functions $\varphi(x)$ representable in the form of a Bessel potential (see ^(1,2))

$$\varphi(x) = (G_1 u)(x) \equiv \int_{R^n} G_1(|x-y|)u(y) dy, \quad u \in L_n(R^n), \quad (1)$$

where $G_1(t)$ is the Bessel kernel of order 1. A vector function $f(x) = (f_1(x), \dots, f_n(x))$ belongs to one of the listed classes if and only if each of its components $f_i(x)$ ($i = 1, \dots, n$) has this property.

Let A be an arbitrary set in R^n . Consider all possible nonnegative functions $u(x) \in L_n(R^n)$ such that for all $x \in A$

$$(G_1 u)(x) \geq 1.$$

The exact lower bound of the quantities $(\|u(x)\|_{L_n(R^n)})$, taken over the set of all such functions $u(x)$, is called the $(1, n)$ -capacity (or simply the capacity) of the set A .

This definition of $(1, n)$ -capacity is a special case of the definition given in ⁽³⁾. In the same work, relations between $(1, n)$ -capacity and other capacities are also indicated.

2. Let B be an open ball in R^n , and let S be its boundary. For an arbitrary vector function $f : B \rightarrow R^n$, set: $E(f)$ is the set of points on S at which the mapping f has angular boundary values; $E'(f)$ is the set of points on S such that for any point $x_0 \in E'(f)$ there exists a path $\gamma_{x_0} \subset B$, tending

to x_0 , along which the mapping f has a limit; $E''(f)$ is the set of points $x_0 \in S$ for each of which there exists a point $y_0 \in \bar{R}^n$ such that

$$\lim_{h \rightarrow 0} \frac{1}{h^n} \int_{B_h(x_0)} |f(x) - y_0| dx = 0, \quad (2)$$

where $B_h(x_0) = \{x \in B : |x - x_0| < h\}$.

Lemma 1. Let $f : B \rightarrow R^n$ be a monotone mapping of the class $W_n^1(B)$, and let $\gamma_{x_0} \subset B$ be an arbitrary path tending to a point $x_0 \in S$. If, for some sequence of points $\{x_n\}$ ($n = 1, 2, \dots$) with the properties

$$x_n \in \gamma_{x_0}, \quad \lim_{n \rightarrow \infty} x_n = x_0, \quad \lim_{n \rightarrow \infty} \frac{\text{diam } x_n \tilde{x}_{n+1}}{\rho(x_n, S)} < 1,$$

where $\rho(x_n, S)$ is the distance from the point x_n to S , and if the limit $\lim_{n \rightarrow \infty} f(x_n) = y_0$ exists, then the mapping has the same limit along the path γ_{x_0} .

The proof is carried out analogously to the proof of Lemma 3 of paper ⁽⁴⁾.

Lemma 2 (see ⁽⁴⁾). Let $f : B \rightarrow R^n$ be a monotone mapping of class $W_n^1(B)$. Then $E(f) = E'(f)$.

Proceeding from condition (2) and applying the preceding lemmas successively, it is not difficult to verify the validity of the following assertion.

Lemma 3. Let $f : B \rightarrow R^n$ be a monotone mapping of class $W_n^1(B)$. Then $E''(f) \subseteq E(f)$.

The following Lemma 4 is a simple consequence of Theorem 5.7 of paper ⁽³⁾ (see also ⁽²⁾).

Lemma 4. Let $f : \dot{R}^n \rightarrow R^n$ be a vector-function of class $L_1^n(R^n)$, and let $f_1 = f|_B$ be its restriction to B . Then the set $S \setminus E'''(f_1)$ is a set of zero capacity.

The following theorem is a generalization of the corresponding results from ⁽⁴⁻⁶⁾.

Theorem 1. Let $f : B \rightarrow R^n$ be a monotone mapping of class $W_n^1(B)$. Then the set $S \setminus E(f)$ is a set of capacity zero.

We give the idea of the proof. By the extension theorem (see, for example, ⁽¹⁾, p. 144) there exists a vector-function $f^* : R^n \rightarrow R^n$ of class $W_n^1(R^n)$ such that $f^*|_B = f$. Further, by Theorem 5.1 of paper ⁽³⁾ (see also ⁽²⁾) there is a vector-function $f^{**} : R^n \rightarrow R^n$ of class $L_1^n(R^n)$, coinciding almost everywhere in R^n with f^* .

Consider $f_1 = f^{**}|_B$. For almost all $x \in B$ we have $f_1(x) = f(x)$, and, consequently, $E''(f_1) = E''(f)$. By Lemma 4 the set $S \setminus E''(f_1) = S \setminus E''(f)$ is a set of capacity zero. Applying Lemma 3, we verify the theorem.

Remark. This theorem can be extended (in the corresponding terms) also to monotone mappings of class $W_p^1(B)$, where $p > n - 1$ (see ⁽⁷⁾).

3. Let $A \subseteq \Omega$ be a set, closed relative to the domain $\Omega \subset R^n$, having no interior points, and let $f : (\Omega \setminus A) \rightarrow R^n$ be a quasiconformal mapping, distinct from the identical constant one. To each point $x_0 \in A$ we assign the following sets of values (see ⁽⁸⁾).

The cluster set $C_{\Omega \setminus A}(f, x_0)$. A value $y \in C_{\Omega \setminus A}(f, x_0)$ if there exists a sequence of points $\{x_n\}$ with the properties

$$x_n \in \Omega \setminus A, \quad \lim_{n \rightarrow \infty} x_n = x_0, \quad \lim_{n \rightarrow \infty} f(x_n) = y.$$

The set of repeated values $R_{\Omega \setminus A}(f, x_0)$. A value $y \in R_{\Omega \setminus A}(f, x_0)$ if there exists a sequence of points $\{x_n\}$ with the properties

$$x_n \in \Omega \setminus A, \quad \lim_{n \rightarrow \infty} x_n = x_0, \quad f(x_n) = y.$$

In paper ⁽¹⁰⁾ we gave certain criteria for the removability of special sets.* The results formulated below concern the structure of the sets $C_{\Omega \setminus A}(f, x_0)$, $R_{\Omega \setminus A}(f, x_0)$ in the case when A is a set of essentially singular points.

Theorem 2. Let $A \subseteq \Omega$ be an arbitrary compact set relative to the domain $\Omega \subset R^n$ of capacity zero. Suppose that $f : (\Omega \setminus A) \rightarrow R^n$ is a quasiconformal mapping (nonconstant) having an essential singularity at each point $x_0 \in A$. Then the cluster set

$$C_{\Omega \setminus A}(f, x_0) = \dot{R}^n.$$

* Taking the opportunity, we note that Theorem 1 formulated in ⁽¹⁰⁾ is meaningful only for $\alpha = n$. For $\alpha > n$ the class of sets of zero α -capacity is empty.

For two-dimensional quasiconformal mappings this assertion was proved in ⁽⁹⁾ (see also ⁽⁸⁾).

Theorem 3. *Let the hypotheses of Theorem 2 be satisfied. Then in any neighborhood of a point $x_0 \in A$ the mapping f assumes infinitely often every value $y \in \overline{R}^n$, with the possible exception of a set of capacity zero; i.e., the set $\overline{R}^n \setminus R_{\Omega \setminus A}(f, x_0)$ is a set of capacity zero.*

In the case where x_0 is an isolated essentially singular point, this theorem is due to Yu. G. Reshetnyak (communication at the Donetsk Colloquium on quasiconformal mappings, 1968).

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CITED LITERATURE

¹ S. M. Nikol'skii, *Approximation of Functions of Several Variables and Imbedding Theorems*, "Nauka," 1969. ² N. Aronszajn, Mulla Fuad, P. Szeptycki, Ann. de l' Inst. Fourier, **13**, 211 (1963). ³ Yu. G. Reshetnyak, Siberian Math. J., **10**, No. 5, 1109 (1969). ⁴ V. M. Miklyukov, Questions of the Geometric Theory of Functions, vol. 4, Tr. Tomsk State Univ., **189**, 80 (1966). ⁵ A. Beurling, Acta Math., **72**, No. 1-2, 1 (1940). ⁶ V. A. Zorich, DAN, **177**, No. 4, 771 (1967). ⁷ V. A. Zhukov, V. M. Miklyukov, Questions of the Geometric Theory of Functions, vol. 5, Tr. Tomsk State Univ., **200**, 88 (1968). ⁸ K. Nosiro, *Cluster Sets*, IL, 1963. ⁹ M. Ohtsuka, Nagoya Math. J., **11**, 131 (1957). ¹⁰ V. M. Miklyukov, DAN, **188**, No. 3, 525 (1969).

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