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Abstract

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MATHEMATICS

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ON SEMIGROUPS HAVING SINGULARITIES AT ZERO SUMMABLE WITH A POWER WEIGHT

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As is known, semigroups of linear continuous operators, strongly continuous only for $t > 0$, may have arbitrary growth as $t \rightarrow 0$. In this note an attempt is made to extend some results of the theory of semigroups of class C_0 to the classes $L^{(n)}$ of semigroups $T(t)$, strongly continuous for $t > 0$, for which the functions $t^n \|T(t)x\|$ ($x \in X$) have summable singularities at zero. Semigroups of this type were first considered by Da Prato ⁽²⁾. Namely, he considered semigroups $T(t)$ satisfying the estimate $t^\alpha \|T(t)\| \leq C$ for some integer α . The results of the present paper make it possible to carry out a finer classification of semigroups, taking into account the case of a fractional exponent α . Moreover, the results obtained make it possible to consider, besides semigroups for which the functions $t^\alpha \|T(t)x\|$ ($x \in X$) are bounded, also semigroups for which the functions $t^\alpha \|T(t)x\|$ are summable. The principal result of the note is the establishment of a correspondence between semigroups of class $L^{(n)}$ and their infinitesimal operators.

1. Suppose that $T(t)$ is a semigroup of linear continuous operators in a Banach space X , strongly continuous for $t > 0$, ω_0 is its type, and A_0 is its infinitesimal operator, i.e. the operator defined by the equality

$$A_0 x = \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \quad (1)$$

on the set $D(A_0)$ of all those $x \in X$ for which the limit (1) exists. Introduce the notation

$$N_0 = \bigcap_{t>0} \{x : T(t)x = 0\}; \quad X_0 = \overline{\bigcup_{t>0} \{T(t)x : x \in X\}}.$$

Theorem 1. *Let $N_0 \cap X_0 = \{0\}$. Then the infinitesimal operator A_0 admits a closure.*

One of the closed extensions of the operator A_0 can be specified in the following way. Define $D(B)$ as the set of all $x \in X$ for which there exists such a $z \in X_0$ that, for all t and s , $0 < s < t$,

$$T(t)x - T(s)x = \int_s^t T(\tau)z d\tau.$$

If the assumptions of Theorem 1 are satisfied, then the element z is determined uniquely by the element x . Therefore the equality $Bx = z$ defines some operator B . This operator is closed and is an extension of the operator A_0 .

Next introduce the operator $R(\lambda)$, defined for $\operatorname{Re} \lambda > \omega_0$ by the formula

$$R(\lambda)x = \lim_{t \rightarrow 0} \int_t^\infty e^{-\lambda s} T(s)x ds \quad (2)$$

on the set D_R of all those $x \in X$ for which the limit (2) exists for arbitrary λ , $\operatorname{Re} \lambda > \omega_0$.

Lemma 1. The operator $R(\lambda)$ maps D_R into D_R for every λ . Moreover, the following relations hold:

- a) $R(\lambda)x - R(\mu)x = (\mu - \lambda)R(\lambda)R(\mu)x \quad (x \in D_R)$;
- b) $(\lambda I - A_0)R(\lambda)x = x$, if

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t T(s)x ds = x;$$

- c) $R(\lambda)(\lambda I - A_0)x = x \quad (x \in D(A_0))$;
- d) $R(\lambda)(\lambda I - B)x = x$, if $x \in D(B)$ and $\lim_{t \rightarrow 0} T(t)x = x$;
- e) $(\lambda I - B)R(\lambda)x = x \quad (x \in D_R \cap X_0)$.

Lemma 2. If $x \in D_R$, then for any nonnegative integer m the equality

$$R^{m+1}(\lambda)x = \frac{1}{m!} \int_0^\infty t^m e^{-\lambda t} T(t)x dt \quad (3)$$

holds.

Let us note that, in contrast to the case of a semigroup of class $L^{(3)}$, the operator $R(\lambda)$ need not be bounded.

We shall say that the semigroup $T(t)$ belongs to the class $L^{(n)}$ if $N_0 = \{0\}$, $X_0 = X$, and if for every $x \in X$ the function $t^n \|T(t)x\|$ is summable on every interval of the form $(0, a)$, $a < \infty$.

Lemma 3. Let $T(t)$ be a semigroup of class $L^{(n)}$. Then the operator $S_n(\lambda)$, defined on X by the formula

$$S_n(\lambda)x = \frac{1}{n!} \int_0^\infty t^n e^{-\lambda t} T(t)x dt \quad (\operatorname{Re} \lambda > \omega_0), \quad (4)$$

is continuous for $\operatorname{Re} \lambda > \omega_0$, and for every $\omega > \omega_0$ the estimate $\|S_n(\lambda)\| \leq M_\omega$ holds ($\operatorname{Re} \lambda > \omega$).

Comparison of formulas (3) and (4) shows that for semigroups of class $L^{(n)}$ the operator $R^{n+1}(\lambda)$, defined on D_R , is bounded, and the operator $S_n(\lambda)$ is its continuous extension to all of X .

For any $x \in X$ the function $S_n(\lambda)x$ is the Laplace transform of the function $t^n(n!)^{-1}T(t)x$, continuous for $t > 0$. Therefore the usual theorems on the Laplace transform are applicable, in particular the uniqueness theorem. In the case of a semigroup of class $L^{(n)}$, it follows from this theorem that $S_n(\lambda)x = 0$ ($\operatorname{Re} \lambda > \omega_0$) if and only if $x = 0$. The same can be said about $R(\lambda)$: $R(\lambda)x = 0$ ($x \in D_R$) if and only if $x = 0$.

2. Let A be a closable operator, and suppose that $D(A^m)$ is dense in X for every m . An analytic function $S_n(\lambda, A)$, defined in some domain $\rho_n(A)$ of the complex plane and taking values in the space of linear continuous operators, will be called the resolvent of order n of the operator A , if from $S_n(\lambda, A)x = 0$ ($\lambda \in \rho_n(A)$) it follows that $x = 0$, and if for $\lambda \in \rho_n(A)$

$$S_n(\lambda, A)Ax = AS_n(\lambda, A)x \quad (x \in D(A));$$

$$S_n(\lambda, A)(\lambda I - A)^{n+1}x = x \quad (x \in D(A^{n+1})). \quad (5)$$

We shall say that the operator A belongs to the class $L_\omega^{(n)}$ if $\rho_n(A) \supseteq \{\lambda : \operatorname{Re} \lambda > \omega\}$, with $\|S_n(\lambda, A)\| \leq M$ ($\operatorname{Re} \lambda > \omega$), and if there exist a nonnegative function $\varphi(t, x)$ ($x \in X$, $t > 0$), continuous jointly in its variables, and a nonnegative function $\varphi(t)$, bounded on every interval of the form (a, b) , $0 < a < b$, for which

$$\lim_{t \rightarrow \infty} t^{-1} \ln \varphi(t) < \omega,$$

that

$$\varphi(t, x) \leq \varphi(t)\|x\|; \quad \int_0^\infty t^n \varphi(t, x) e^{-\omega t} dt < \infty, \quad (6)$$

$$\|S_n^{(m)}(\tau, A)x\| \leq \frac{1}{n!} \int_0^\infty e^{-\tau t} t^{n+m} \varphi(t, x) dt \quad (\tau > \omega, m = 0, 1, \dots). \quad (7)$$

Theorem 2. *The infinitesimal operator A_0 of a semigroup $T(t)$ of class $L^{(n)}$ belongs to the class $L_\omega^{(n)}$ for every $\omega > \omega_0$, and $S_n(\lambda, A_0) = S_n(\lambda)$ ($\operatorname{Re} \lambda \geq \omega$).*

Theorem 3. *If A is an operator of class $L_\omega^{(n)}$, then there exists a unique semigroup $T(t)$ of class $L^{(n)}$ whose infinitesimal operator A_0 satisfies the relation $S_n(\lambda, A_0) = S_n(\lambda, A)$. Moreover $\omega_0 \leq \omega$, $\|T(t)x\| \leq \varphi(t, x)$, and $\|T(t)\| \leq \varphi(t)$.*

Let the operator A have, for $\operatorname{Re} \lambda \geq \omega$, a resolvent $S_n(\lambda, A)$ of order n , and let there exist a family $\{T(t) : t > 0\}$ of linear continuous operators, strongly continuous in t , such that

$$\int_0^\infty t^n e^{-\omega_1 t} \|T(t)x\| dt < \infty$$

for some n and ω_1 and for every x . Suppose, in addition, that

$$S_n(\lambda, A)x = \frac{1}{n!} \int_0^\infty t^n e^{-\lambda t} T(t)x dt \quad (\operatorname{Re} \lambda > \max(\omega, \omega_1)).$$

Then it follows from Theorem 3 that $T(t)$ is a semigroup of class $L^{(n)}$.

If we fix the functions $\varphi(t, x)$ and $\varphi(t)$, then the estimates (7) give us a necessary and sufficient condition for the function $S_n(\tau, A)$ to correspond to a semigroup $T(t)$ satisfying the estimates $\|T(t)x\| \leq \varphi(t, x)$ and $\|T(t)\| \leq \varphi(t)$. By choosing various functions $\varphi(t, x)$ and $\varphi(t)$, we shall obtain conditions for a semigroup to belong to one or another class of semigroups. In the case $n = 0$ this path is considered in detail in ⁽³⁾.

It is natural to supplement Theorem 3 with the following assertion.

Theorem 4. *Let a closed operator A belong to the class $L_\omega^{(n)}$ and have a resolvent $R(\lambda, A)$ for $\operatorname{Re} \lambda > \omega$. Then the closure $\bar{A}_0$ of the operator A_0 coincides with the operator A , and $S_n(\lambda, A) = R^{n+1}(\lambda, A)$.*

3. We shall call a semigroup $T(t)$ of class $L^{(n)}$ a semigroup of class $L_0^{(n)}$ if the function $t^n \|T(t)\|$ is summable on every interval of the form $(0, a)$, $a < \infty$. Obviously, if $T(t)$ is a semigroup of class $L_0^{(n)}$, then the estimates

$$\|S_n^{(m)}(\lambda)\| \leq \frac{1}{n!} \int_0^\infty t^{n+m} e^{-t \operatorname{Re} \lambda} \|T(t)\| dt \leq \frac{1}{n!} \int_0^\infty t^{n+m} e^{-t \operatorname{Re} \lambda} \varphi(t) dt,$$

are valid if $\|T(t)\| \leq \varphi(t)$ and the function $t^n \varphi(t) e^{-\omega t}$ is summable on $(0, \infty)$. The converse assertion is also valid:

Theorem 5. *Let an operator A have a resolvent $S_n(\lambda, A)$ of order n for real $\tau > \omega$, and suppose the estimates*

$$\|S_n^{(m)}(\tau, A)\| \leq \frac{1}{n!} \int_0^\infty t^{n+m} e^{-\tau t} \varphi(t) dt \quad (\tau > \omega, m = 0, 1, \dots), \quad (8)$$

where the function $\varphi(t)$ is nonnegative and $t^n \varphi(t) e^{-\omega t}$ is summable on $(0, \infty)$. Then there exists a semigroup $T(t)$ of class $L_0^{(n)}$ for which $S_n(\tau) = S_n(\tau, A)$ for $\tau > 0$.

4. As an application, let us consider semigroups $T(t)$ with $N_0 = \{0\}$ and $X_0 = X$, satisfying the estimate $\|T(t)\| \leq C t^{-\alpha} e^{\omega t}$. We shall call these semigroups semigroups of class C_α . They belong to the classes $L_0^{(n)}$ for $n \geq [\alpha]$. The estimates (8) take for them a simpler form.

Theorem 6. Let $T(t)$ be a semigroup of class C_α . Then its infinitesimal operator A_0 has, for $\operatorname{Re} \lambda > \omega$, a resolvent $S_n(\lambda, A_0)$ of order n ($n \geq [\alpha]$), and

$$\|S_n^{(m)}(\lambda, A_0)\| \leq \frac{C \Gamma(m+n+1-\alpha)}{n! (\operatorname{Re} \lambda - \omega)^{m+n+1-\alpha}} \quad (\operatorname{Re} \lambda > \omega, m = 0, 1, \dots). \quad (9)$$

Conversely, suppose an operator A has, for real $\tau > \omega$, a resolvent $S_n(\tau, A)$ of order n , and the estimate

$$\|S_n^{(m)}(\tau, A)\| \leq \frac{C \Gamma(m+n+1-\alpha)}{n! (\tau - \omega)^{m+n+1-\alpha}} \quad (\tau > \omega, m = 0, 1, \dots) \quad (10)$$

holds. Then there exists a unique semigroup $T(t)$ of class $L_0^{(n)}$, satisfying the estimate $\|T(t)\| \leq C t^{-\alpha} e^{\omega t}$, for which

$$S_n(\tau, A)x = \frac{1}{n!} \int_0^\infty t^n e^{-\tau t} T(t)x dt.$$

Theorem 6 is a generalization of the Da Prato theorem ⁽²⁾ to the case of fractional α . Another theorem of a similar kind for semigroups satisfying the estimate $\|T(t)\| \leq C t^{-\alpha} e^{\omega t}$ was proved by P. E. Sobolevskii ⁽⁴⁾.

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