

TWO EXAMPLES IN THE THEORY OF DIMENSION OF BICOMPACTA

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Abstract

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MATHEMATICS

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TWO EXAMPLES IN THE THEORY OF DIMENSION OF BICOMPACTA

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In the present note two bicompecta are constructed. In § 1 a completely normal bicompectum Z is constructed for which $\dim Z = 1$, $\text{ind } Z = \text{Ind } Z = 2$. In § 2 a bicompectum Φ with the first axiom of countability is constructed for which $\dim \Phi = 1$, $\text{ind } \Phi = \text{Ind } \Phi = 2$, and moreover the small inductive dimension is equal to 2 at every point of the bicompectum Φ .

In constructing the bicompectum Z one has to use Suslin's hypothesis on the existence of an ordered, completely normal, nonseparable connected bicompectum, as well as the continuum hypothesis, which asserts that $2^{\aleph_0} = \aleph_1$. In Cohen's paper ⁽²⁾ it is proved that the continuum hypothesis may be adopted as an axiom of set theory. Tennenbaum ⁽³⁾ proved the same thing with respect to Suslin's hypothesis.

§ 1. 1. In what follows it is more convenient for us to deal with an ordered, connected, completely normal nonseparable bicompectum S , no open subset of which is separable. From the existence of an arbitrary Suslin continuum there follows the existence of the required one.

2. The product $Z^* = S \times I$ of the Suslin continuum with the ordinary interval, as is easily seen, is a completely normal bicompectum, and therefore its weight and the cardinality of the set of its closed subsets do not exceed $2^{\aleph_0}$ ⁽¹⁾.
3. In Z^* one can choose a subset everywhere dense in Z^* such that its intersection with any nowhere dense closed subset is at most countable.

This can be done as follows. Number by transfinite ordinals smaller than ω_1 the set of all nowhere dense closed subsets $\{F_\alpha\}$, $\alpha < \omega_1$, of the bicompectum Z^* , and some base $\{U_\alpha\}$, $\alpha < \omega_1$, of cardinality $\aleph_1 = 2^{\aleph_0}$ of the bicompectum Z . For each $\alpha < \omega_1$ take such a point z_α^* that $z_\alpha^* \notin \bigcup_{\beta \leq \alpha} F_\beta$ and $z_\alpha^* \in U_\alpha$. Such a point z_α^* can be chosen, since in the bicompectum Z^* no open set is representable as the union of nowhere dense sets. We may assume that $z_\alpha^* \notin S \times 1$ and $z_\alpha^* \in S \times 0$ for every $\alpha < \omega_1$. The set $M = \{z_\alpha^*, \alpha < \omega_1\}$ of the selected points has the required property (this is nothing other than Lusin's ν -set).

4. For every point z^* of the bicom pactum Z^* there exists a monotone mapping f_{z^*} of the bicom pactum Z^* onto the square I^2 which is one-to-one at the point z^* , i.e., the set $f_{z^*}^{-1}f_{z^*}(z^*)$ consists of one point, and such that the preimage of a base at the point $f_{z^*}(z^*)$ will be a base at the point $z^* \in Z^*$.
5. Let the point z^* belong to the set M . Consider the mapping f_{z^*} of the bicom pactum Z^* onto the square I^2 . We insert in Z^* and I^2 , instead of the points z^* and $f_{z^*}(z^*)$, circles. For the points of the circle inserted in the square I^2 , neighborhoods are defined in the same way as this is done in ⁽⁴⁾. Let $f_{z^*}(z^*) = y$. Points of the circle inserted instead of the point y will be denoted by (y, φ) . As a neighborhood of the point (y, φ) we take an arc of the circle

$((y, \varphi_1), (y, \varphi_2))$, containing our point (y, φ) and a part of a circular sector of some radius on the square, enclosed between the rays issuing from the point y at the angles φ_1 and φ_2 . For all the remaining points of the square the neighborhoods are the previous ones. For the points of the circle (z^*, φ) inserted into the bicom pactum Z^* , we define neighborhoods as the inverse images of the neighborhoods of the corresponding points on the square. It is easily verified that the space thus obtained is again a bicom pactum.

Now, for each point of the set M , instead of it, in an analogous way we insert a circle and likewise define the neighborhoods of points on these circles. Moreover, if, in defining a neighborhood for some point, a point from the set M falls into the neighborhood, then we assume that the whole circle by which this point is replaced also belongs to the neighborhood being defined. The newly obtained space Z is a perfectly normal bicom pactum. The proof of the last assertion is analogous to that given in the work ⁴.

6. The natural mapping π of the bicom pactum Z onto the bicom pactum Z^* , which glues the inserted circles back into points, is monotone and irreducible.

By a horizontal (vertical) in the bicom pactum Z we shall mean any bicom pact set B whose image under the mapping π is a horizontal (vertical) in the bicom pactum Z^* . Moreover, if $\pi(B)$ contains a point z^* of the set M , then the set B must contain only the points $(z^*, 0)$ and (z^*, π) ($(z^*, 1/2\pi)$ and $(z^*, 3/2\pi)$) from the corresponding circle, and we shall say that the horizontal (vertical) B passes through the indicated circle.

7. We shall show that $\text{Ind } Z = 2$. It is clear that $\text{Ind } Z \leq 2$, and it remains to prove that $\text{Ind } Z \geq 2$. Let A be any nowhere dense partition between the sets $S \times 1$ and $S \times 0$ in the bicom pactum Z . The set $\pi(A)$ is a nowhere dense partition between the sets $S \times 1$ and $S \times 0$ in the bicom pactum Z^* ; therefore it contains no more than a countable number of points $z_1^*, z_2^*, \dots, z_n^*, \dots$ of the set M . At the remaining points of the set $\pi(A)$ the mapping π is a homeomorphism. Let $s_1, s_2, \dots, s_n, \dots$ be the projections of the points $z_1^*, z_2^*, \dots, z_n^*, \dots$ on the bicom pactum S . Since the continuum S is not

separable, there is in it an interval $(\tilde{s}_1, \tilde{s}_2)$ containing none of the points $s_1, s_2, \dots, s_n, \dots$. Consider the bicom pactum $[\tilde{s}_1, \tilde{s}_2] \times I = Z_1^*$. The set $\pi(A) \cap Z_1^*$ is a partition between the sets $[\tilde{s}_1, \tilde{s}_2] \times 1$ and $[\tilde{s}_1, \tilde{s}_2] \times 0$ in the bicom pactum Z_1^* . By standard arguments one can show that the set $\pi(A) \cap Z_1^*$ contains a one-dimensional bicom pactum, and since the mapping π is a homeomorphism on this set, a one-dimensional bicom pactum is also contained in the set A . Thus, $\text{Ind } Z \geq 2$.

Since every open subset of the continuum S is nonseparable, analogous reasoning shows that at each point z of the bicom pactum Z the small inductive dimension is equal to 2, i.e. $\text{ind}_z Z = 2$.

8. We shall prove that $\dim Z = 1$. Let $\omega = \{U_1, U_2, \dots, U_p\}$ be an arbitrary finite open covering of the bicom pactum Z . The number of circles $\{K_1, K_2, \dots, K_r\}$, each of which does not belong wholly to any element of the covering, is finite; otherwise, taking on each circle a point and choosing a convergent subsequence $z_1^*, z_2^*, \dots, z_n^*, \dots$, we would obtain that the element of the covering ω containing the limiting point contains almost all the indicated circles. We inscribe into the original covering ω an open covering $\gamma = \{V_1, V_2, \dots, V_m\}$ such that the stars of the circles $K_1, K_2, \dots, K_r$ do not intersect. We inscribe into the covering the circle K_i ($i = 1, 2, \dots, r$), cut out by the covering γ , a finite refinement of multiplicity 2 and extend it to a refinement of multiplicity 2 of some neighborhood O_i of this circle. Take a rectangle Π_i ($i = 1, 2, \dots, r$) with sides parallel to the factors, which contains the circle K_i and is contained in its neighborhood O_i . We shall now draw horizontals and verticals, including also the extensions of the sides of the recta-

angles Π_i ($i = 1, 2, \dots, r$), so that the partition determined by them is inscribed in the covering γ ; moreover, we discard those parts of these vertical and horizontal lines that intersect the rectangles Π_i ($i = 1, 2, \dots, r$), and on the rectangles Π_i ($i = 1, 2, \dots, r$) we take the partition previously constructed there. As a result we obtain a partition $\nu = \{D_1, D_2, \dots, D_l\}$, which in general has multiplicity 4, and on the rectangles Π_i ($i = 1, 2, \dots, r$) has multiplicity 2. Let the points at which the partition has multiplicity 3 or 4 be $z_1, z_2, \dots, z_{n_1}$. Suppose that the point z_1 belongs to four elements of the partition $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{i_4} . The boundaries of these elements of the partition form a "cross" at the point z_1 , i.e. a certain piece of a vertical and a horizontal line.

Take a rectangle $\tilde{\Pi}_1$ containing only the point z_1 and belonging to the intersection of elements of the covering γ which contain the elements of the partition $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{i_4} . In the lower right corner of the rectangle $\tilde{\Pi}_1$ choose a circle and draw through it a vertical and a horizontal line up to their intersection with the boundary of the rectangle $\tilde{\Pi}_1$. Then $\tilde{\Pi}_1$ is divided into 4 rectangles: $\tilde{\Pi}_1^1$ —the upper left, $\tilde{\Pi}_1^2$ —the upper right, $\tilde{\Pi}_1^3$ —the lower left, $\tilde{\Pi}_1^4$ —the lower right. To the upper left element of the partition D_{i_1} we add the rectangle $\tilde{\Pi}_1^1$, i.e.

$$\widetilde{D}_{i_1} = \left(D_{i_1} \setminus \bigcup_{i=2}^4 \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^1.$$

Instead of the upper right element D_{i_2} of the partition ν we take

$$\widetilde{D}_{i_2} = \left(D_{i_2} \setminus \bigcup_{\substack{i=1 \\ i \neq 2}}^4 \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^2,$$

instead of the lower left element D_{i_4} we take

$$\widetilde{D}_{i_4} = \left(D_{i_4} \setminus \bigcup_{i=1}^3 \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^4,$$

and instead of the lower right element D_{i_3} we take the set

$$\widetilde{D}_{i_3} = \left(D_{i_3} \setminus \bigcup_{\substack{i=1 \\ i \neq 3}}^4 \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^3.$$

Now, instead of the elements of the partition $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{i_4} , we have the elements of the partition $\widetilde{D}_{i_1}, \widetilde{D}_{i_2}, \widetilde{D}_{i_3}$ and $\widetilde{D}_{i_4}$ of multiplicity 2, and no new points at which the multiplicity is ≥ 3 have appeared. Thus, carrying out a finite number of rearrangements of the partition ν at the points $z_1, z_2, \dots, z_{n_1}$, we obtain a partition of multiplicity 2 inscribed in the covering γ , and consequently also in the initial covering ω . This proves our assertion.

§ 2. 1. Consider the set of countable sequences $\{a_1, a_2, \dots\}$ of numbers of the interval $[0, 1]$, and take only those sequences in which, if a 0 or a 1 occurs at some place, then the same number stands in all subsequent places. The topology on the set of all such sequences is introduced by lexicographic ordering of them: $\{a_i, i = 1, 2, \dots\} < \{\beta_i, i = 1, 2, \dots\}$ if the first number at which these sequences differ is smaller in the first than in the second. After introducing the topology we obtain an ordered continuum L with the first axiom of countability. The set B of sequences in which 0 or 1 occurs is everywhere dense in L . Every interval of the continuum L contains a continuum homeomorphic to it. Therefore the continuum L can be represented as follows: in the interval $[0, 1]$ each point, except for the endpoints, is replaced by the continuum L .

2. Let the bicomactum Φ^* be the product of the continuum and the interval $I = [0, 1]$. The interval $(0, 1)$ can be represented as the sum of a continuum number of everywhere dense mutually disjoint sets P^ξ , $\xi \in \Omega$. Renumber all points $\langle l^\xi \rangle$, $\xi \in \Omega$, of the continuum L in whose notation as sequences of

numbers neither 0 nor 1 occurs. Take on the segment $I^\xi \times I$ the set $I^\xi \times P^\xi$. Denote the sum of all such sets by D . This set is everywhere dense in the bicom pactum Φ^* . Note that any horizontal line in the bicom pactum Φ^* contains only one point of the set D , with the exception of the horizontal lines $L \times 0$ and $L \times 1$.

3. Just as in the preceding paragraph, in the bicom pactum Z^* replace each point d of the set D by a circle and define neighborhoods of points

of these neighborhoods by means of mutually one-to-one mappings of the bicom pactum Φ^* onto the square I^2 at the corresponding point d . We obtain a bicom pactum Φ . The natural mapping π of the bicom pactum Φ onto the bicom pactum Φ^* is continuous, monotone, and irreducible.

4. At every point y of the bicom pactum Φ we have

$$\text{ind}_y \Phi = 2.$$

Indeed, let Oy be an arbitrary neighborhood of the point y . We may assume that this neighborhood does not meet the lower or upper base of the bicom pactum Φ . We shall suppose that the open set Oy does not meet the upper base. Take in the set Oy a point $y_1 = (l_1, \alpha_1)$ of the bicom pactum Φ , whose coordinate l_1 belongs to the set B of the continuum L . The horizontal $L \times \alpha_1$ meets only one circle, and therefore one can take on it a point $y_2 = (l_2, \alpha_1)$ so that the part of the horizontal $[(l_2, \alpha_1), (l_1, \alpha_1)]$ does not meet the circle and belongs to the neighborhood Oy , and the point y_2 may be chosen so that the coordinate l_2 also belongs to the set B . Consider the rectangle

$$\Pi = [l_2, l_1] \times [\alpha_1, 1]$$

in the bicom pactum Φ^* and its inverse image $\pi^{-1}(\Pi) = P$ in the bicom pactum Φ . The set $\text{Fr}(Oy) \cap P = A$ is a nowhere dense partition between the upper and lower bases in the bicom pactum P .

We shall show that $\text{ind } A \geq 1$. The set $\pi(A)$ is a nowhere dense partition between the upper and lower bases in the rectangle Π . We shall suppose that $\pi(A)$ is connected. Let

$$\Pi \setminus \pi(A) = V_1 \cup V_2,$$

where V_1 and V_2 are open, $V_1 \cap V_2 = \emptyset$, and V_1 contains the top, while V_2 contains the bottom of the rectangle Π . The mapping π is a homeomorphism on the set of points of one-to-one correspondence, and it is non-one-to-one only at the points $\{d\}$ of the set D of the bicom pactum Φ^* . Consequently, it is enough to show that the set $\pi(A)$ contains a connected bicom pactum not containing points of the set D . If the set $\pi(A)$ contains a horizontal segment, then in it there is a smaller segment not containing points of the set D . Suppose now that the set $\pi(A)$ contains no horizontal segment. Represent the continuum $[l_2, l_1]$ of the bicom pactum Φ^* in the same way as the continuum L : in the segment $[0, 1]$ each point t is replaced by a continuum $\tilde{L}$ homeomorphic to L . As the points

t of the segment $[0, 1]$ take points from the set B . The continuum $\tilde{L}$, inserted in place of the point t , will be denoted by $\tilde{L}_t$. Denote the projections of the bicom pactum Π onto its factors by π_1 and π_2 , respectively. For each rectangle $\tilde{L}_t \times [0, 1]$ put

$$\alpha_1(t) = \sup\{\pi_2(y), y \in (\tilde{L}_t \times [0, 1]) \cap \pi(A)\},$$

$$\alpha_2(t) = \min\{\pi_2(y), y \in (\tilde{L}_t \times [0, 1]) \cap \pi(A)\},$$

with $\alpha_1(t) > \alpha_2(t)$. Each segment

$$\tilde{L}_t \times \alpha \quad (\alpha_1(t) < \alpha < \alpha_2(t))$$

meets the sets V_1 and V_2 . Taking for each t the segment $[\alpha_1(t), \alpha_2(t)]$, we obtain an uncountable set of segments. Since the segment $[0, 1]$ has a countable base, there is a segment $[h_1, h_2]$ contained in an infinite set of segments $[\alpha_1(t_i), \alpha_2(t_i)]$, $i = 1, 2, \dots$. Let the sequence $t_1, t_2, \dots$ converge to the point $\hat{t}$ (otherwise we choose a convergent subsequence). Then the segment

$$[(\hat{t}, h_1), (\hat{t}, h_2)]$$

belongs to the set $\pi(A)$, since every neighborhood of this segment contains points of the sets V_1 and V_2 . Since the segments $t \times [0, 1]$ do not contain points of the set D , this proves that $\text{Ind } A \geq 1$. Consequently, it has been proved that $\text{ind } \Phi \geq 2$. It is evident that

$$\text{ind } \Phi \leq \text{Ind } \Phi \leq 2.$$

Thus,

$$\text{ind } \Phi = \text{Ind } \Phi = 2.$$

5. The fact that

$$\dim \Phi = 1$$

is proved in exactly the same way as for the bicom pactum Z in § 2.

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Note: Figure translations are in progress. See original paper for figures.

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