

# SOME INTEGRAL RELATIONS IN THE THEORY OF RADIATION TRANSFER

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**Abstract**

**Full Text**

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*MATHEMATICAL PHYSICS*

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## **SOME INTEGRAL RELATIONS IN THE THEORY OF RADIATION TRANSFER**

*(Presented by Academician A. N. Tikhonov on 8 IV 1970)*

In Theorem 1 new integral relations are given which connect various solutions of Milne's problem.

The mathematical problem is posed as follows. Let  $g(\mu)$ ,  $\mu \in [-1, 1]$ , be a scattering indicatrix, that is, a nonnegative function on  $[-1, 1]$  with summable square and such that

$$0 < g_0 \leq 1.$$

(Here and below

$$g_n = 2\pi \int_{-1}^1 g(\mu) P_n(\mu) d\mu, \quad n \geq 0;$$

$P_n(\mu)$  is the  $n$ -th Legendre polynomial.)

Let  $\Omega$  be the unit sphere of three-dimensional space,  $C(\Omega)$  the space of continuous and  $L_2(\Omega)$ -square-summable complex-valued functions on  $\Omega$ ; in  $L_2(\Omega)$  the scalar product is defined by

$$(f, g) = \int_{\Omega} f(\omega) \overline{g(\omega)} d\omega$$

and the norm by  $\|\cdot\| = \sqrt{(\cdot, \cdot)}$ .

$g(\mu)$  generates a linear completely continuous operator  $\hat{g} : L_2(\Omega) \rightarrow C(\Omega)$ , acting according to the formula

$$\hat{g}f(\omega) \equiv (\hat{g}f)(\omega) = \int_{\Omega} g(\omega\omega') f(\omega') d\omega', \quad f \in L_2(\Omega).$$

Next let  $\mathcal{B}_\varepsilon$  be the class of abstract functions  $x(\tau)$  of the numerical variable  $\tau \in (0, \infty)$ , possessing the following properties.

- a) For each  $\tau \in (0, \infty)$ ,  $x(\tau)$  is an element of  $L_2(\Omega)$ .
- b) The numerical function  $\|x(\tau)\|$  is bounded on every bounded part of the half-line  $\tau > 0$ .
- c) The abstract function  $e^{-\varepsilon\tau}x(\tau)$  is summable in the sense of Bochner <sup>(3)</sup> on  $(0, \infty)$ .

Fix some vector  $\mathbf{n} \in \Omega$ . Let  $\varepsilon \in (-1, 1)$ ,  $x(\cdot) \in \mathcal{B}_\varepsilon$ , and

$$\hat{g}x(\tau, \omega) \equiv (\hat{g}(x(\tau)))(\omega), \quad \tau > 0, \quad \omega \in \Omega.$$

It turns out that for any fixed  $\omega \in \Omega$  the function  $e^{-\tau}\hat{g}x(\tau, \omega)$  is summable with respect to the variable  $\tau$  on  $(0, \infty)$ . Hence it follows that the following formula uniquely defines the operator  $\hat{A}$ , acting on  $x(\cdot)$ :

$$(\hat{A}x(\cdot))(\tau, \omega) = \begin{cases} \frac{1}{\omega\mathbf{n}} \int_0^\tau e^{-(\tau-\rho)/\omega\mathbf{n}} \hat{g}x(\rho, \omega) d\rho, & \text{for } \omega\mathbf{n} > 0, \\ \hat{g}x(\tau, \omega), & \text{for } \omega\mathbf{n} = 0, \\ \frac{1}{|\omega\mathbf{n}|} \int_\tau^\infty e^{-(\rho-\tau)/|\omega\mathbf{n}|} \hat{g}x(\rho, \omega) d\rho, & \text{for } \omega\mathbf{n} < 0. \end{cases}$$

It turns out that the classes  $\mathcal{B}_\varepsilon$ ,  $\varepsilon \in (-1, 1)$ , are invariant with respect to the transformation  $\hat{A}$ : if  $x(\cdot) \in \mathcal{B}_\varepsilon$ , then also  $\hat{A}x(\cdot) \in \mathcal{B}_\varepsilon$ .

The Milne problem is the problem of solving the equation

$$x(\cdot) = \hat{A}x(\cdot) + f(\cdot) \tag{1}$$

in the class  $\overline{\mathfrak{B}} = \bigcup_{\varepsilon \in (-1, 1)} \mathfrak{B}_\varepsilon$ . Here we shall be interested in the so-called Milne problem with insolation, corresponding to equation (1) with a free term  $f$  of the special form:

$$f(\tau, \omega) = B(\omega) \exp(-\tau/\omega n), \quad \tau > 0, \quad \omega \in \Omega; \tag{2}$$

$$B \in L_2(\Omega), \quad B(\omega) = 0 \quad \text{when } \omega n \leq 0. \tag{3}$$

We shall call problem 1 the problem (1)–(3). Problem 1 with  $B \equiv 0$  is the classical homogeneous Milne problem. (Physical interpretation: problem 1 is the problem of the diffusion of radiation through the half-space  $\tau > 0$  of three-dimensional space;  $n$  is the unit normal vector to the boundary of the half-space,

directed inward;  $B(\omega)$  is the angular distribution of the  $\omega$ -radiation incident on the boundary from outside.)

In the theory of the Milne problem the characteristic equation plays a fundamental role

$$(1 + k\omega n)\psi(\omega) = g\psi(\omega). \quad (4)$$

Here  $k$  is a complex parameter,  $k \in Z_0$ , where  $Z_0$  is the complex plane with cuts along the real axis from  $-\infty$  to  $-1$  and from  $1$  to  $+\infty$  (the points  $\pm 1$  belong to the cuts).

Denote by  $\mathfrak{N}$  the set of values of the parameter  $k \in Z_0$  for which (4) admits a nontrivial solution  $\psi \in L_2(\Omega)$ . It turns out that  $\mathfrak{N}$  lies entirely in the interval  $(-1, 1)$  of the real axis,  $\mathfrak{N}$  is symmetric with respect to the point  $k = 0$ ,  $\mathfrak{N}$  is nonempty and at most countable. If  $\mathfrak{N}$  is infinite, then the points  $k = \pm 1$  are limit points of  $\mathfrak{N}$ , and  $\mathfrak{N}$  has no other limit points.

There exists exactly one  $\lambda_0 \in \mathfrak{N}$ ,  $\lambda_0 \geq 0$ , such that for  $k = \lambda_0$  (4) admits a sign-constant solution. Up to normalization this solution is unique. If  $g_0 = 1$ , then  $\lambda_0 = 0$ ; if  $g_0 < 1$ , then  $\lambda_0 > 0$ . The interval  $(-\lambda_0, \lambda_0)$  contains no points of the set  $\mathfrak{N}$ .

For  $k \in \mathfrak{N}$  the eigenspace of (4) has finite dimension  $p_k$ , with  $p_{-k} = p_k$ ,  $k \in \mathfrak{N}$ , and  $p_{\lambda_0} = 1$ . If  $\mathfrak{N}$  contains points different from zero, then there exists a system of functions

$$\{\psi_{kp} \mid k \in \mathfrak{N}, k \neq 0, p = 1, 2, \dots, p_k\} \quad (5)$$

such that

- a)  $\psi_{kp}(\omega)$  is real,  $\psi_{kp} \in C(\Omega)$ ,  $p = 1, 2, \dots, p_k$ ,  $k \in \mathfrak{N}$ ,  $k \neq 0$ ;
- b)  $((\omega n)\psi_{kp}, \psi_{kp'}) = -\delta_{p'p} \operatorname{sgn} k$  for  $1 \leq p \leq p' \leq p_k$ ,  $k \in \mathfrak{N}$ ,  $k \neq 0$ ;
- c)  $\psi_{(-k)p}(\omega) = \psi_{kp}(-\omega)$ ,  $\omega \in \Omega$ ,  $p = 1 \div p_k$ ,  $k \in \mathfrak{N}$ ,  $k \neq 0$ ;
- d)  $\{\psi_{kp} \mid p = 1, 2, \dots, p_k\}$  is a basis of the eigenspace of equation (4) corresponding to the given value of the parameter  $k \in \mathfrak{N}$ ,  $k \neq 0$ .

A system of functions (5) possessing the listed properties is not unique. We assume that, from all such systems, one has been chosen and fixed; moreover (in the case  $g_0 < 1$ ) this has been done so that  $\psi_{\lambda_0,1}(\omega) > 0$  everywhere on  $\Omega$  (such a choice is possible).

Let  $x(\cdot)$  be some solution of problem 1,  $x(\cdot) \in \mathfrak{B}_\varepsilon$ ,  $\varepsilon \in (-1, 1)$ . Then there exists  $x(0) \in L_2(\Omega)$  such that

- a)  $x(0, \omega) = B(\omega)$  when  $\omega n > 0$ ;

$$\text{b) } \lim_{\tau \rightarrow 0+0} \|x(\tau) - x(0)\| = 0.$$

The assertions stated above are justified in <sup>(1)</sup>.

We now single out two variants  $1_{1,2}$  of problem 1, corresponding to two fixed functions  $B = B_{1,2}$  in formulas (1)–(3). Let  $\varepsilon_j \in (-1, 1)$ ,

and  $x_j(\cdot) \in \mathcal{B}_\varepsilon^j$  are some fixed solutions of problems  $1_j$ ,  $j = 1, 2$ . Put

$$D = \begin{cases} \frac{3}{4\pi}(1 - g_1)(x_1(0), \omega n)(x_2(0), \omega n), & \text{if } g_0 = 1; \\ 0, & \text{if } g_0 < 1; \end{cases}$$

and, for each  $\lambda \in \mathfrak{R}$ ,

$$A_\lambda = \begin{cases} \frac{3}{4\pi}[(x_1(0), \omega n)(x_2(0), (\omega n)^2) - (x_2(0), \omega n)(x_1(0), (\omega n)^2)], & \text{for } \lambda = 0; \\ \operatorname{sgn} \lambda \cdot \sum_{p=1}^{p_\lambda} (x_1(0), (\omega n)\psi_{\lambda p})(x_2(0), (\omega n)\psi_{(-\lambda)p}), & \text{for } \lambda \neq 0. \end{cases}$$

Finally, introduce the operator  $\hat{\sigma} : L_2(\Omega) \rightarrow L_2(\Omega)$  by the formula  $(\hat{\sigma}f)(\omega) = f(-\omega)$ ,  $f \in L_2(\Omega)$ .

**Theorem 1.** For all  $\tau \geq 0$ ,

$$(x_1(\tau), (\omega n) \hat{\sigma} \overline{x_2(0)}) = D\tau + \sum_{\lambda \in \mathfrak{R} \cap (-\varepsilon_2, \varepsilon_1)} A_\lambda e^{\lambda \tau}. \quad (6)$$

(The bar over the top denotes complex conjugation; the interval  $(-\varepsilon_2, \varepsilon_1)$  is the empty set if  $\varepsilon_1 + \varepsilon_2 \leq 0$ .)

Formula (6) makes it possible to establish certain new connections between the solutions of Milne's problem (both homogeneous and with insolation) and, in particular, to express explicitly the coefficients of the asymptotic expansions of the solutions of problem 1 (cf. (1)) in terms of the law of insolation  $B$  and the boundary values (for  $\tau = 0$ ) of the solutions of the homogeneous Milne problem. Expressions of this type were first obtained by V. V. Sobolev <sup>(2)</sup>.

A detailed exposition of the proof of Theorem 1 and of its consequences is contained in <sup>(4)</sup>.

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