

DUALITY FOR COHERENT ANALYTIC SHEAVES

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Abstract

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MATHEMATICS

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DUALITY FOR COHERENT ANALYTIC SHEAVES

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1. Let X be a complex analytic manifold, countable at infinity, of complex dimension n , and let F be a coherent analytic sheaf on X . Each point of the manifold X has a holomorphically complete open neighborhood U , over which there is defined an epimorphism of sheaves

$$\pi : O^m \rightarrow F,$$

where O is the sheaf of germs of holomorphic functions on X , and m is some positive integer. Consequently, the complex vector space $\Gamma(U; F)$ of all continuous sections of the sheaf F over U is isomorphic to the quotient space $\Gamma(U; O^m)/\Gamma(U; R)$, where R is the kernel of the epimorphism π . Endowing the space $\Gamma(U; O^m)$ with the topology of compact convergence, we thereby introduce on $\Gamma(U; F)$ a certain separated locally convex topology, which is a Fréchet topology and does not depend on the choice of the epimorphism π (cf. ⁽¹⁾).

Let $\mathfrak{U} = (U_i)$ be a sufficiently fine locally finite covering of the manifold X by holomorphically complete open sets. The space $C^k(\mathfrak{U}; F)$ of cochains of degree $k \geq 0$ of the covering $\mathfrak{U}$ with coefficients in F is endowed with the product topology of the spaces $\Gamma(U_{i_0} \cap \dots \cap U_{i_k}; F)$. Further, the space $H^k(\mathfrak{U}; F)$ of cohomology of the covering $\mathfrak{U}$ is endowed with the quotient-space topology $\text{Ker } \delta_k / \text{Im } \delta_{k-1}$, where δ_k is the (continuous) coboundary operator. Finally, the space $H^k(X; F)$ of Čech cohomology of the manifold X with coefficients in the sheaf F is endowed with the topology of the inductive limit of the spaces $H^k(\mathfrak{U}; F)$ with respect to the filtering set of classes of pairwise equivalent coverings. By means of a diagram chase analogous to the proof of A. Weil's de Rham theorem (see ⁽²⁾), it is easy to show that the canonical mapping

$$H^k(\mathfrak{U}; F) \rightarrow H^k(X; F)$$

is an isomorphism of topological vector spaces.

2. Let E^k , for any integer $k \geq 0$, be the sheaf of germs of infinitely differentiable exterior differential forms of bidegree $(0, k)$ on the manifold X . Consider the complex $\Gamma(X; E^* \otimes_O F)$ of vector spaces $\Gamma(X; E^k \otimes_O F)$ ($k = 0, 1, \dots$), whose coboundary operator

$$d_k'' : \Gamma(X; E^k \otimes_O F) \rightarrow \Gamma(X; E^{k+1} \otimes_O F)$$

is induced by the exterior differential $d'' : E^k \rightarrow E^{k+1}$. For each $U \in \mathfrak{U}$, endow the vector space $\Gamma(U; E^k \otimes_O F)$ with the strongest of the locally convex topologies under which the canonical bilinear mapping

$$\Gamma(U; E^k) \times \Gamma(U; F) \rightarrow \Gamma(U; E^k \otimes_O F),$$

which assigns to sections ω and ξ over U of the sheaves E^k and F , respectively, the section $x \mapsto \omega(x) \otimes \xi(x)$ of the sheaf $E^k \otimes_O F$. The space $\Gamma(X; E^k \otimes_O F)$ is endowed with the wea-

weakest of the topologies for which the restriction mappings are continuous

$$\Gamma(X; E^k \otimes_O F) \rightarrow \Gamma(U; E^k \otimes_O F) \quad (U \in \mathfrak{u}).$$

Finally, the cohomology space $H^k(\Gamma(X; E^* \otimes_O F))$ of the complex $\Gamma(X; E^* \otimes_O F)$ is endowed with the topology of the quotient space $\text{Ker } d_k'' / \text{Im } d_{k-1}''$.

Since E_x^k is a flat $\mathcal{O}_x$ -module for every $x \in X$ (see (3)), by the Dolbeault-Grothendieck lemma the sequence of sheaves

$$0 \rightarrow F \rightarrow \mathcal{O}_p \otimes_O F \rightarrow E^1 \otimes_O F \rightarrow \dots$$

is exact. Consequently, by means of a diagram chase one can show that, for every $k \geq 0$, the topological vector spaces $H^k(X; F)$ and $H^k(\Gamma(X; E^* \otimes_O F))$ are canonically isomorphic.

3. Let a be a continuous linear form on the space $\Gamma(X; E^k \otimes_O F)$. If U is an arbitrary open set in X , then for any $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$ we have a section $x \mapsto \omega(x) \otimes \xi(x)$ over X of the sheaf $E^k \otimes_O F$, continuous in ω and ξ . Fixing ξ , we obtain a continuous linear form $\omega \mapsto a(\omega \otimes \xi)$ on the space $\Gamma_c(U; E^k)$ of sections with compact supports, endowed with the usual topology. Thereby a $\Gamma(U; \mathcal{O})$ -linear mapping $\varphi_U : \Gamma(U; F) \rightarrow \Gamma(U; D^{n, n-k})$ is defined, where $D^{n, n-k}$ is the sheaf of germs of currents of bidegree $(n, n-k)$ on X , and

$$\langle \omega, \varphi_U(\xi) \rangle = a(\omega \otimes \xi)$$

for $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$. It is obvious that, for $V \subset U$, the mappings φ_V and φ_U satisfy the natural compatibility conditions, i.e. the family (φ_U)

determines a certain homomorphism of $\mathcal{O}$ -modules $\varphi : F \rightarrow D^{n,n-k}$. Since the form a has compact support, the homomorphism φ also has compact support. Thus a canonical linear mapping of vector spaces is defined

$$\Gamma'(X; E^k \otimes_{\mathcal{O}} F) \rightarrow \Gamma_c(X; \text{Hom}_{\mathcal{O}}(F, D^{n,n-k})),$$

which assigns to a continuous linear form a on $\Gamma(X; E^k \otimes_{\mathcal{O}} F)$ a homomorphism $\varphi : F \rightarrow D^{n,n-k}$ with compact support.

If $\varphi = 0$, then $a = 0$. Indeed, with the help of a suitable refinement of the identity it is easy to verify that, on each compact subset of X , any global section of the sheaf $E^k \otimes_{\mathcal{O}} F$ is representable as a finite sum of sections $\omega \otimes \xi$, where $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$ for some open U , and $a(\omega \otimes \xi) = \langle \omega, \varphi_U(\xi) \rangle = 0$. Consequently, the mapping $a \mapsto \varphi$ of the vector space $\Gamma'(X; E^k \otimes_{\mathcal{O}} F)$ into $\Gamma_c(X; \text{Hom}_{\mathcal{O}}(F, D^{n,n-k}))$ is injective. We show that it is also surjective. Since the homomorphism φ has compact support, it suffices to define the corresponding form a on sections of the form $\omega \otimes \xi$, with $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$, by the formula $a(\omega \otimes \xi) = \langle \omega, \varphi_U(\xi) \rangle$. It is obvious that the form a so defined is continuous on $\Gamma(X; E^k \otimes_{\mathcal{O}} F)$, since each of the modules $\Gamma(U; F)$ may be regarded as having a finite number of generators over the ring $\Gamma(U; \mathcal{O})$ (for which it is enough to choose U sufficiently small and holomorphically complete).

Thus the vector spaces $\Gamma'(X; E^k \otimes_{\mathcal{O}} F)$ and $\Gamma_c(X; \text{Hom}_{\mathcal{O}}(F, D^{n,n-k}))$ are canonically isomorphic.

Consider the mapping

$${}^t d'' : \Gamma_c(X; \text{Hom}_{\mathcal{O}}(F, D^{n,n-k})) \rightarrow \Gamma_c(X; \text{Hom}_{\mathcal{O}}(F, D^{n,n-k})),$$

identified with the mapping conjugate to

$$d'' : \Gamma(X; E^k \otimes_{\mathcal{O}} F) \rightarrow \Gamma(X; E^{k+1} \otimes_{\mathcal{O}} F).$$

For any sections $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$ we have:

$$\langle \omega, {}^t d'' \varphi_U(\xi) \rangle = {}^t d'' a(\omega \otimes \xi) = a(d'' \omega \otimes \xi) = (-1)^{k+1} \langle \omega, d'' \varphi_U(\xi) \rangle,$$

i.e., the mapping d'' is induced by the sheaf homomorphism

$$(-1)^{k+1} d'' : D^{n,n-k-1} \rightarrow D^{n,n-k}.$$

4. Let $D^{n,k}$, for $k \geq 0$, be the sheaf of germs of currents of bidegree (n, k) on the manifold X . The corresponding Dolbeault-Grothendieck resolution of the sheaf Ω^n of germs of holomorphic differential forms of degree n on X can be embedded in the commutative diagram

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \Omega^n & \rightarrow & D^{n,0} & \rightarrow & D^{n,1} & \rightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & L^0 & \rightarrow & L^{0,0} & \rightarrow & L^{0,1} & \rightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & L^1 & \rightarrow & L^{1,0} & \rightarrow & L^{1,1} & \rightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow &
\end{array}$$

with exact rows and columns and with injective modules $L^p, L^{p,q}$ ($p, q \geq 0$) over the sheaf of rings O . Since the sheaf F is coherent, $\text{Hom}_O(F, G)_x \approx \text{Hom}_{O_x}(F_x, G_x)$ for any O -module G (see (4)). On the other hand, from Malgrange's theorem (3) it follows that $D_x^{n,k}$ is an injective O_x -module for every $x \in X$. Consequently, there is a commutative diagram

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \Gamma_c(X; \text{Hom}_O(F, \Omega^n)) & \rightarrow & \Gamma_c(X; \text{Hom}_O(F, D^{n,0})) & \rightarrow & \Gamma_c(X; \text{Hom}_O(F, D^{n,1})) & \rightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \Gamma_c(X; \text{Hom}_O(F, L^0)) & \rightarrow & \Gamma_c(X; \text{Hom}_O(F, L^{0,0})) & \rightarrow & \Gamma_c(X; \text{Hom}_O(F, L^{0,1})) & \rightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \Gamma_c(X; \text{Hom}_O(F, L^1)) & \rightarrow & \Gamma_c(X; \text{Hom}_O(F, L^{1,0})) & \rightarrow & \Gamma_c(X; \text{Hom}_O(F, L^{1,1})) & \rightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow &
\end{array}$$

in which all rows and columns, beginning with the second, are exact. By means of a diagram chase we obtain a canonical isomorphism of the cohomology vector spaces, respectively, of the complexes $\Gamma_c(X; \text{Hom}_O(F, L^{*,*}))$ and $\Gamma_c(X; \text{Hom}_O(F, D^{n,*}))$.

Consider the covariant functor

$$L \rightarrow \text{Hom}_{O,c}(X; F, L) = \Gamma_c(X; \text{Hom}_O(F, L)),$$

defined on the category of O -modules and taking values in the category of vector spaces. We shall denote its derived functors by $\text{Ext}_{O,c}^k(X; F, L)$ ($k = 0, 1, \dots$). Thus, it has been proved above that for every $k \geq 0$ the vector spaces $\text{Ext}_{O,c}^k(X; F, \Omega^n)$ and $H^k(\Gamma_c(X; \text{Hom}_O(F, D^{n,*})))$ are canonically isomorphic.

We endow the space $\Gamma_c(X; \text{Hom}_O(F, D^{n,k}))$ with the strong topology with respect to $\Gamma(X; E^{n-k} \otimes_O F)$. Then in the space $\text{Ext}_{O,c}^k(X; F, \Omega^n)$ it is natural to define the topology canonically identified with the topology of the quotient space $\text{Ker } d_k'' / \text{Im } d_{k-1}''$, where

$$d_k'' : \Gamma_c(X; \text{Hom}_O(F, D^{n,k})) \rightarrow \Gamma_c(X; \text{Hom}_O(F, D^{n,k+1}))$$

is the coboundary operator of the complex $\Gamma_c(X; \text{Hom}_O(F, D^{n,*}))$. We shall denote by $\widetilde{\text{Ext}}_{O,c}^k(X; F, \Omega^n)$ the associated separated locally convex space, i.e., the quotient space of $\text{Ext}_{O,c}^k(X; F, \Omega^n)$ by the closure of zero in it.

5. Since the topological vector space $H^k(X; F)$ is canonically identifiable with the quotient space $\text{Ker } d_k'' / \text{Im } d_{k-1}''$, where

$$d_k'' : \Gamma(X; E^k \otimes_O F) \rightarrow \Gamma(X; E^{k+1} \otimes_O F),$$

the dual space $(H^k(X; F))'$ may be identified with the quotient space

$$(\text{Im } d_{k-1}'')^\circ / (\text{Ker } d_k'')^\circ,$$

where the polars are taken in

$$\Gamma_c(X; \text{Hom}_O(F, D^{n,n-k})).$$

Since

$$\langle d_{k-1}'' \omega, \varphi \rangle = \langle \omega^t, d_{k-1}'' \varphi \rangle$$

for all $\omega \in \Gamma(X; E^{k-1} \otimes_O F)$, $\varphi \in \Gamma_c(X; \text{Hom}_O(F, D^{n,n-k}))$, it follows from $\langle d_{k-1}'' \omega, \varphi \rangle = 0$ (for every ω) that ${}^t d_{k-1}'' \varphi = 0$, and conversely. In other words, $(\text{Im } d_{k-1}'')^\circ$ coincides with

$$\text{Ker } {}^t d'' : \Gamma_c(X; \text{Hom}_O(F, D^{n,n-k})) \rightarrow \Gamma_c(X; \text{Hom}_O(F, D^{n,n-k+1})).$$

Similarly, the equality

$$\langle \omega, {}^t d_k'' \varphi \rangle = 0$$

(for every φ) is equivalent to the equality $d_k'' \omega = 0$, i.e. $\text{Ker } d_k''$ coincides with

$$(d'' \Gamma_c(X; \text{Hom}_O(F, D^{n,n-k-1})))^\circ.$$

In this case, $(\text{Ker } d_k'')^\circ$ is the closure in the weak topology of the subspace

$$d'' \Gamma_c(X; \text{Hom}_O(F, D^{n,n-k-1}))$$

in

$$\Gamma_c(X; \text{Hom}_O(F, D^{n,n-k})).$$

Since $\Gamma(X; E^k \otimes_O F)$ is reflexive, the weak and strong closures of subspaces in

$$\Gamma_c(X; \text{Hom}_O(F, D^{n,n-k}))$$

coincide. Thus we have proved

Theorem 1. For $0 \leq k \leq n$, the vector space

$$\widetilde{\text{Ext}}_{O,c}^{n-k}(X; F, \Omega^n)$$

is canonically identifiable with the space dual to the topological vector space $H^k(X; F)$.

If the mapping

$$d_k'' : \Gamma(X; E^k \otimes_O F) \rightarrow \Gamma(X; E^{k+1} \otimes_O F)$$

is a homomorphism, then the dual mapping

$${}^t d_k'' : \Gamma_c(X; \text{Hom}_O(F, D^{n, n-k-1})) \rightarrow \Gamma_c(X; \text{Hom}_O(F, D^{n, n-k}))$$

has closed image. In this case the space

$$\widetilde{\text{Ext}}_{O, c}^{n-k}(X; F, \Omega^n)$$

is Hausdorff and, consequently, coincides with

$$\text{Ext}_{O, c}^{n-k}(X; F, \Omega^n).$$

The mapping d_k'' is a homomorphism if the space $H^{k+1}(X; F)$ is finite-dimensional (see ⁽⁵⁾). Thus the following theorem holds; for (not necessarily) projective varieties it was first proved by Grothendieck ⁽⁶⁾.

Theorem 2. If the variety X is compact, then the vector space

$$\text{Ext}_O^{n-k}(X; F, \Omega^n)$$

is canonically identifiable with the dual of the space $H^k(X; F)$.

If the sheaf F is locally isomorphic to the sheaf $\mathcal{O}^m$, then

$$\text{Ext}_{O, c}^{n-k}(X; F, \Omega^n) \approx H_c^{n-k}(X; \text{Hom}_O(F, \Omega^n)).$$

In particular, if $F = \Omega^p(V)$ is the sheaf of germs of holomorphic differential forms of degree p with values in the complex vector bundle V , then

$$\text{Hom}_O(F, \Omega^n) \approx \Omega^{n-p}(V'),$$

where V' is the vector bundle dual to V . Consequently, we have

Theorem 3. The vector space

$$H_c^{n-k}(X; \Omega^{n-p}(V'))$$

is canonically identifiable with the dual of the topological vector space

$$H^k(X; \Omega^p(V)).$$

In the particular case when

$$d_k'' : \Gamma(X; E^{p, k}(V)) \rightarrow \Gamma(X; E^{p, k+1}(V))$$

is a homomorphism, we obtain from this the result of J.-P. Serre ⁽⁵⁾.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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