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I. A. Ibragimov, R. Z. Khas'minskii

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Abstract

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MATHEMATICS

I. A. IBRAGIMOV, R. Z. KHASHMINSKII

ON THE ASYMPTOTIC BEHAVIOR OF GENERALIZED BAYES ESTIMATES

(Presented by Academician A. N. Kolmogorov on 11 III 1970)

1. Let $x_1, \dots, x_n$ be a sample with independent elements from a population with distribution P_θ , depending on an unknown parameter $\theta \in \Theta$. Suppose that P_θ , for all θ , are distributions in R_n , and that the parameter set $\Theta \subset R_m$. Suppose also that all measures P_θ are absolutely continuous with respect to some measure ν , and put $f(x, \theta) = dP_\theta/d\nu$.

Our aim will be to study the asymptotic behavior of the finite-dimensional distributions of the random field

$$p_n(\theta) = \prod_1^n f(x_i, \theta) \pi(\theta) / \int \prod_1^n f(x_i, \theta) \pi(\theta) \nu(d\theta) \quad (1)$$

and of estimates $\tilde{\theta}_n$ of the parameter θ , having the form

$$\tilde{\theta}_n = \int \theta p_n(\theta) \nu(d\theta) \quad (2)$$

for measurable functions $\pi(\theta)$ in R_m . If $\pi(\theta)$ is the density of a probability distribution in R_m with respect to the measure ν , then the field $p_n(\theta)$ and the estimate $\tilde{\theta}_n$ are naturally interpreted as the Bayesian posterior density and, respectively, the posterior mean of a random variable having the prior distribution with density $\pi(\theta)$. We shall retain the names posterior density and Bayesian estimate for the right-hand sides of (1) and (2), although in our considerations θ is not a random variable. We shall denote by θ_0 the true (unknown to the observer) value of the parameter θ .

We shall call a function $\varphi(n)$ a correct normalizing factor (c.n.f.) for (1), (2), if the finite-dimensional distributions of the random field $\eta_n(\theta) = \varphi^{-1}(n) p_n(\theta_0 + \theta/\varphi(n))$ and the distribution of the random variable $\zeta_n = \varphi(n)(\tilde{\theta}_n - \theta_0)$, as $n \rightarrow \infty$, converge to proper limiting distributions.

In the present note we are concerned with finding c.n.f.'s and the corresponding limiting distributions under various restrictions imposed on the family P_θ . To make the exposition less cumbersome, we shall restrict ourselves to the case where the random variables x_i and the parameter θ are one-dimensional, and $\nu(dy) = dy$, Lebesgue measure.*

2. Let $f(x, \theta)$ be a measurable function of (x, θ) , and let the "prior density" $\pi(\theta)$ be continuous at the point θ_0 , bounded on compact sets, and

$$\iint |\theta| f(x, \theta) f(x, \theta_0) |\pi(\theta)| dx d\theta < \infty.$$

With respect to the family P_θ , below one of the following conditions will be assumed to hold:

$$\text{I. } 1) \lim_{|\theta| \rightarrow \infty} \int f(x, \theta) f(x, \theta_0) dx = 0; \quad 2) \lim_{\varepsilon \rightarrow 0} \int f^{1-\varepsilon}(x, \theta_0) dx = 1.$$

* We owe the formulation of some of the problems considered here to Yu. V. Linnik.

- II. Whatever the real numbers $\theta_1, \dots, \theta_s$, there exist intervals $[0, T_1], \dots, [0, T_s]$ such that, for all $t_j \in [0, T_j]$, as $\xi \downarrow 0$,

$$\int \prod_{j=1}^s \left(\frac{f(x, \theta_0 + \xi \theta_j)}{f(x, \theta_0)} \right)^{t_j} f(x, \theta_0) dx = 1 + \xi^\alpha a(t_1, \dots, t_s; \theta_1, \dots, \theta_s) + o(\xi^\alpha), \quad (3)$$

where the number α does not depend on $\theta_1, \dots, \theta_s, t_1, \dots, t_s$, and the function a is not identically zero.

- III. For all θ_1, θ_2 with $|\theta_1|, |\theta_2| \leq H < \infty$, as $\xi \rightarrow 0$,

$$\left| \int \ln \frac{f(x, \theta_0 + \xi \theta_1)}{f(x, \theta_0 + \xi \theta_2)} f(x, \theta_0 + \xi \theta_2) dx \right| \leq |\xi|^\alpha C_1(|\theta_2 - \theta_1|),$$

$$\left| \int \ln^2 \frac{f(x, \theta_0 + \xi \theta_1)}{f(x, \theta_0 + \xi \theta_2)} f(x, \theta_0 + \xi \theta_2) dx \right| \leq |\xi|^\alpha C_2(|\theta_2 - \theta_1|);$$

here the functions $C_i(h)$, $i = 1, 2$, (depending on H) tend to zero as $h \rightarrow 0$, and in both integrals the function $f(x, \theta_0 + \xi \theta_1)$ is to be replaced by one wherever it is equal to zero.

Denote by $Y(\theta)$ a (separable) random process whose finite-dimensional distributions are determined by the equalities

$$M \exp\{\Sigma t_{jY}(\theta_j)\} = \exp\{a(t_1, \dots, t_s; \theta_1, \dots, \theta_s)\},$$

where here and below $M(\cdot) = M_{\theta_0}(\cdot)$, $P(\cdot) = P_{\theta_0}(\cdot)$.

Theorem 1. Suppose that assumptions I-III are satisfied. Then, for n.h.m. $\varphi(n) = n^{1/\alpha}$, the finite-dimensional distributions of the process $\eta_n(\theta) = \varphi^{-1}(n)p_n(\theta_0 + \theta/\varphi(n))$, $\theta \in (-\infty, +\infty)$, converge to the corresponding distributions of the process

$$\eta_0(\theta) = \left(\int e^{Y(\theta)} d\theta \right)^{-1} e^{Y(\theta)},$$

and the normalized differences $\xi_n = \varphi(n)(\tilde{\theta}_n - \theta_0)$ converge in distribution to the random variable

$$\int \theta \eta_0(\theta) d\theta.$$

3. If, for example, the left-hand side of (3) can be differentiated twice under the integral sign, we find that $\alpha = 2$,

$$\eta_\xi(\theta) = \frac{1}{\sqrt{2\pi I}} \exp \left\{ - \left(\theta - \frac{\xi}{\sqrt{I}} \right)^2 / 2I \right\}; \quad Y(\theta) = \sqrt{I} \theta \xi - \frac{\theta^2}{2} I, \quad (4)$$

where ξ is a standard normal random variable, and I is the Fisher information. Therefore the normalized posterior density is asymptotically Gaussian, and the difference $\sqrt{n}(\tilde{\theta}_n - \theta_0)$ is also asymptotically normal with parameters $0, I^{-1/2}$ (see also item 5). Under similar assumptions one can also prove that $\tilde{\theta}_n$ is close to the maximum likelihood estimate $\hat{\theta}_n$ in the sense that $\sqrt{n}(\tilde{\theta}_n - \hat{\theta}_n) \rightarrow 0$ in probability, or with probability 1, depending on the restrictions on P_θ . We shall not dwell on these results in greater detail, because closely related theorems are available in the literature (see, for example, ⁽¹⁾, where one can also find the history of the question going back to S. N. Bernstein and R. Mises).

4. Let us turn to discontinuous densities $f(x, \theta)$. We shall assume that the function $f(x, \theta)$ is bounded, continuously differentiable with respect to θ on each of the intervals $(x_k(\theta), x_{k+1}(\theta))$, and has discontinuities of the first kind at the points $x_k(\theta)$, $k = 1, \dots, r$. Suppose that the functions $x_k(\theta)$ are continuously differentiable, and denote

$$p(k) = \lim_{x \downarrow x_k(\theta_0)} f(x, \theta_0), \quad q(k) = \lim_{x \uparrow x_k(\theta_0)} f(x, \theta_0).$$

Consider two independent Poisson random measures concentrated on the set $R = \{1, 2, \dots, r\}$. It will be convenient for us to denote these measures by the common symbol $\nu(\theta, A)$, $A \subseteq R$. This will not cause misunderstandings,

since for one of these measures the parameter $\theta \in (-\infty, 0]$, and for the other $\theta \in [0, \infty)$, $\nu(\theta, A) = 0$. The distributions of the measures $\nu(\theta, A)$ are determined by the equalities: $M\nu(\theta, k) = p(k)x'_k(\theta_0)\theta$, if $\theta x'_k(\theta_0) > 0$, and $M\nu(\theta, k) = -q(k)x'_k(\theta_0)\theta$, if $\theta x'_k(\theta_0) < 0$. Denote by R^* the subset of points k in R for which $p(k)q(k) = 0$, and put

$$\tau_+ = \inf_{\theta > 0} \{\theta : \nu(\theta, R^*) > 0\}, \quad \tau_- = \sup_{\theta < 0} \{\theta : \nu(\theta, R^*) > 0\}$$

(if R^* is empty, we shall take $\tau_+ = \infty$, $\tau_- = -\infty$).

Theorem 2. Suppose that, in addition to the assumptions of item 4, the assumptions of item 2 are satisfied and $|f'_\theta(x, \theta)| \leq q(x)$, where $\int q(x) dx < \infty$. Then the conclusion of Theorem 1 is valid with $r(n) = n$ and with limiting process

$$Y(\theta) = \theta \sum_{k=1}^r (p(k) - q(k))x'_k(\theta_0) + \int_R \text{sign}(x'_z(\theta_0)\theta) \ln \frac{q(z)}{p(z)} \nu(\theta, dz)$$

$$\text{for } \theta \in (\tau_-, \tau_+), \quad Y(\theta) = -\infty \quad \text{for } \theta \notin [\tau_-, \tau_+).$$

Moreover, the limiting distribution for $n(\bar{\theta} - \theta_0)$ can be written in the form

$$\left(\int_{\tau_-}^{\tau_+} \exp Y(\theta) d\theta \right)^{-1} \int_{\tau_-}^{\tau_+} \theta \exp Y(\theta) d\theta.$$

The conditions of this theorem are satisfied, for example, by the cases considered in (2), when: 1) $f(x, \theta) = \beta$ for $0 \leq x \leq \theta$, $f(x, \theta) = \gamma$ for $\theta < x \leq 1$, and $f(x, \theta) = 0$ for $x \notin [0, 1]$; 2) $f(x, \theta)$ has one point of discontinuity.

5. In this section we shall assume that $f(x, \theta) = f(x - \theta)$, so that the parameter to be estimated is a shift parameter. We also put $\pi(\theta) \equiv 1$. The estimator (2) is now the Pitman estimator for the shift parameter (see (3)); without loss of generality one may put $\theta_0 = 0$. Suppose, moreover, that

$$\int |x|f(x) dx < \infty. \quad (5)$$

Theorem 3. Suppose condition (5) is satisfied, the function $f(x)$ is absolutely continuous, and the Fisher information

$$I = \int \frac{|f'(x)|^2}{f(x)} dx$$

is finite. Then the finite-dimensional distributions of the process

$$\eta_n(\theta) = n^{-1/2} p_n(\theta_0 + \theta/\sqrt{n})$$

converge to the corresponding distributions of the process $\eta_0(\theta)$ defined by formula (4), and the difference $\sqrt{n}(\bar{\theta}_n - \theta_0)$ is asymptotically normal with parameters 0, $I^{-1/2}$.

Theorem 4. Suppose the conditions of Theorem 3 are satisfied and $M|x_i|^p < \infty$ for some $p > 1$. Then

$$\lim_{n \rightarrow \infty} n^{p/2} M|\bar{\theta}_n - \theta_0|^p = I^{-p/2} \pi^{-1/2} 2^{(p+1)/2} \Gamma((p+1)/2). \quad (6)$$

By imposing further restrictions on the moments of the random variables $l^{(k)}(x_i)$, where $l(x) = \ln f(x)$, one can obtain further terms of the asymptotic expansion of the left-hand side of (6). For example, suppose the conditions of Theorem 3 are satisfied and, in addition, the moments are finite:

$$M|l^{(k)}(x_i)|^s, \quad k = 1, \dots, 5, \quad s \leq 20; \quad M \left(\max_{|\theta| \leq \varepsilon} |l^{(5)}(x_i - \theta)| \right)^s, \quad s \leq 20.$$

Then

$$M(\bar{\theta}_n - \theta_0)^2 = \frac{1}{In} + \frac{C_1}{n^2} + o(n^{-2}).$$

Similarly, if $\tilde{\theta}_n$ is the general estimate (2), but with $\pi(\theta) \equiv 1$, $\nu(dy) \equiv dy$, and the moments $M|l_{\theta}^{(k)}(x_i, \theta_0)|^s$, $k = 1, \dots, 5$; $s \leq 20$, are finite, as is

$$M \max_{|\theta| \leq \varepsilon} |l_{\theta}^{(5)}(x_0, \theta_0 - \theta)|^s, \quad s \leq 20,$$

where $l(x, \theta) = \ln f(x, \theta)$, and if $\max M\tilde{\theta}_n^2 < \infty$, then

$$M(\tilde{\theta}_n - \theta_0) = \frac{c(\theta_0)}{n} + o\left(\frac{1}{n}\right); \quad c(\theta_0) = \frac{1}{I^2} [Ml_{\theta}'''(x_i, \theta_0) + Ml'(x_i, \theta_0)l''(x_i, \theta_0)],$$

$$M(\tilde{\theta}_n - \theta_0)^2 = \frac{1}{In} + \frac{c_1(\theta_0)}{n^2} + o\left(\frac{1}{n^2}\right), \quad I = M|l'(x_i, \theta_0)|^2.$$

6. In order to determine what happens when the regularity conditions are violated, consider the following class of densities $f(x, \theta) = f(x - \theta)$. The function $f(x)$ is twice continuously differentiable in each of the intervals (x_k, x_{k+1}) , $k = 0, \dots, r$, $x_0 = -\infty$, $x_{r+1} = \infty$, and, as $x \rightarrow x_k$, $k = 1, \dots, r$, its derivative $f'(x)$ satisfies the conditions:

$$f'(x) \sim a_k^- \alpha_k^- |x - x_k|^{\alpha_k^- - 1}, \quad x \rightarrow x_k - 0; \quad f'(x) \sim a_k^+ \alpha_k^+ |x - x_k|^{\alpha_k^+ - 1},$$

$$x \rightarrow x_k + 0,$$

where $-1 < \alpha_k^\pm < 1$, $\alpha_k^\pm \neq 0$. It will be convenient for us also to include here the case $\alpha_k^\pm = 0$, assuming that the equality $\alpha_k^+ = 0$ ($\alpha_k^- = 0$) means that $f(x_k + 0) = a_k^+$ ($f(x_k - 0) = a_k^-$).

Introduce the notation

$$\gamma = \min_k (\alpha_k^+, \alpha_k^-), \quad A^\pm = \{k : \alpha_k^\pm = \gamma\}, \quad C = \{k : f(x_k) = 0\},$$

$$\gamma_1 = \min\{2 \min_{k \in C} \alpha_k^\pm, \min_{k \in C} \alpha_k^\pm\}, \quad A_1^\pm = \{k : k \in C, 2\alpha_k^\pm = \gamma_1\},$$

$$A_2^\pm = \{k : k \in C, \alpha_k^\pm = \gamma_1\}.$$

The following assertions are valid:

- 1) If $\gamma \leq 0$, then the n.n.m. $\varphi(n) = n^{1/(1+\gamma)}$; in this case the only parameters determining the limiting distribution for $\eta_n(\theta)$ and ξ_n are the numbers γ and $\alpha_k^\pm$ for $k \in A^\pm$.
- 2) If $\gamma > 0$, then the n.n.m. $\varphi(n) = n^{1/(1+\gamma_1)}$, and the only parameters determining the limiting distributions for $\eta_n(\theta)$, ξ_n are the numbers γ_1 and $\alpha_k^\pm$ for $k \in A_1^\pm \cup A_2^\pm$.

These assertions make it possible to order the singularities of the function $f(x)$ according to their “informativeness,” if one restricts oneself to the class of singularities of power type. They can also be extended to the case of singularities of the form $|x - x_k|^\alpha l(x - x_k)$, where $l(x)$ is a slowly varying function in the sense of Karamata (already in (3) the factor ξ^α may be replaced by $\xi^{\alpha l(\xi)}$). It is interesting to note that slowly varying functions in the n.n.m. may also appear at the “extreme” points of power singularities of the function $f(x)$. For example, for a function $f(x)$ having only singularities of the form $a_k |x - x_k|$ and $c_k + a_k |x - x_k|^{1/2}$, $c_k > 0$, the n.n.m. is $\varphi(n) = \sqrt{n \ln n}$.

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Leningrad State University
named after A. A. Zhdanov

Institute for Problems of Information Transmission
Academy of Sciences of the USSR
Moscow

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Note: Figure translations are in progress. See original paper for figures.

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